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Economic Situations of Lagrange Multiplier When Costs of Various Inputs Increase for Nonlinear Budget Constraint

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Abstract

An industry always expects to survive in profit maximization atmosphere. To develop profit maximization strategy, it must follow scientific methods in every step of production and distribution. Only proper decisions can propel the industry smoothly in sustainable way. This study attempts to discuss economic effects of Lagrange multiplier when per unit costs of various inputs increase. In this paper the method of Lagrange multiplier is applied to represent higher dimensional unconstrained problem from the lower dimensional constrained problem. Cobb-Douglas production function, 6×6 bordered Hessian matrix, and 6×6 Jacobian are operated here to provide economic predictions appropriately. In the study profit maximization is considered with subject to the nonlinear budget constraint.

Keywords: Profit maximization, Lagrange multiplier, nonlinear budget constraint

1. Introduction

Modern economics cannot forward smoothly without mathematical modeling (Samuelson, 1947). Mathematics helps the industries for the development of their global, regional and national financial structure (Ferdous & Mohajan, 2022). At present world mathematical modeling becomes a leading discipline of many fields of social sciences, such as economics, sociology, psychology, political science, etc. (Carter, 2001). In economics it is used to solve optimization problems, problems of demand and supply, etc. (Samuelson, 1947). An industry always wants its own benefits and also sees the welfare of the society (Eaton & Lipsey, 1975).

Profit maximization practice is essential for the sustainability of an industry (Islam et al. 2010). In multivariable calculus, the method of Lagrange multiplier is a very useful and powerful technique (Baxley & Moorhouse, 1984). In this study we have used Cobb-Douglas production function, the determinant of 6×6 bordered Hessian matrix, 6×6 Jacobian, and four input variables to provide economic predictions (Cobb & Douglas, 1928). Throughout the study we have shown mathematical calculation in some details but in simple forms.

2. Literature Review

In any type of research, literature review is an introductory section, where works of previous researchers are included (Polit & Hungler, 2013). Two American professors Charles W. Cobb (1875-1949) and Paul H. Douglas (1892-1976) for the first time have developed a production function, which is known as Cobb-Douglas production function (Cobb & Douglas, 1928). Another two American professors John V. Baxley and John C. Moorhouse have worked on the optimization problem, such as profit maximization (Baxley & Moorhouse, 1984). Since then, many scholars worked on profit maximization using mathematical devices. Professor Jamal Nazrul Islam (1939-2013), a well-known mathematician of Bangladesh, and his coauthors have elaborately worked on profit maximization (Islam et al., 2010, 2011). On the other hand, Cambodian Professor Pahlaj Moolio and his coworkers have considered the Cobb-Douglas production functions in their study to analyze the mathematical structure of profit maximization (Moolio et al., 2009).

Devajit Mohajan and Haradhan Kumar Mohajan have scrutinized the profit maximization, utility maximization, and cost minimization, where they have discussed sensitivity analyses (Mohajan & Mohajan, 2022a-g, 2023a-g). Jannatul Ferdous and Haradhan Kumar Mohajan in their study have considered three inputs, such as capital, labor, and other inputs for the mathematical analysis of the production procedures of the industry (Ferdous & Mohajan, 2022). Lia Roy and her coauthors have developed a series of theorems during cost minimization studies (Roy et al., 2021).

3. Research Methodology of the Study

Research is a creative work that needs systematic investigations. To do a good research a researcher should be a devotee in collection, interpretation and refinement of data (Pandey & Pandey, 2015). Methodology is a guideline for the accomplishment of a good research (Kothari, 2008). Therefore, research methodology is the specific procedures that are used to identify, select, process, and analyze materials (Somekh & Lewin, 2005).

In this study we have worked on Cobb-Douglas production function. We have also worked on the determinant of 5×5 bordered Hessian and 5×5 Jacobian matrices (Mohajan, 2017a, 2018a, 2020). With the mathematical properties of calculus, we have depended on the secondary data sources of profit maximization, such as journal articles, books, etc. (Mohajan, 2017b, 2018b). In the study we have shown the mathematical analyses in some details.

4. Objective of the Study

The chief objective of this study is to discuss the economic situations of Lagrange multiplier when the costs of various inputs are increased. Other minor objectives of the study are as follows:

- to show the mathematical calculations in some details, and
- to provide the economic results properly.

5. Lagrangian Function

We consider that an industry tries to make a maximum profit from its products, and it wants to establish a sustainable environment in the economic world. Let the industry uses A_1 amount of capital, A_2 quantity of labor, A_3 quantity of principal raw materials, and A_4 quantity of irregular raw material for its usual production process. Let us consider the Cobb-Douglas production function $f(A_1, A_2, A_3, A_4)$ as a profit function for our economic model (Cobb & Douglas, 1928; Islam et al., 2010; Mohajan & Mohajan, 2023a),

$$P(A_1, A_2, A_3, A_4) = f(A_1, A_2, A_3, A_4) = \Gamma A_1^{\alpha} A_2^{\beta} A_3^{\gamma} A_4^{\delta}, \tag{1}$$

where Γ is the efficiency parameter that reflects the level of technology, i.e., technical process, economic system, etc., which represents total factor productivity. Moreover, Γ reflects the skill and efficient level of the workforce. Here α, β, γ, and δ are parameters; α indicates the output of elasticity of capital, and measures the percentage change in $P(A_1, A_2, A_3, A_4)$ for 1% change in A_1, while A_2, A_3, and A_4 are held constants. Similarly, β indicates the output of elasticity of labor, γ indicates the output of elasticity of principal raw materials, and δ indicates the output of elasticity of irregular raw material. Now these four parameters α, β, γ, and δ must satisfy the following four inequalities (Islam et al., 2011; Moolio et al., 2009; Mohajan, 2022; Mohajan & Mohajan, 2022a):

$$0 < \alpha < 1,\ 0 < \beta < 1,\ 0 < \gamma < 1, \text{ and } 0 < \delta < 1. \tag{2}$$

A strict Cobb-Douglas production function, in which $\Psi = \alpha + \beta + \gamma + \delta < 1$ indicates decreasing returns to scale, $\Psi = 1$ indicates constant returns to scale, and $\Psi > 1$ indicates increasing returns to scale. Now we consider that the profit function is subject to a nonlinear budget constraint as (Roy et al., 2021; Mohajan & Mohajan, 2022b, 2023b),

$$B(A_1, A_2, A_3, A_4) = kA_1 + lA_2 + mA_3 + n(A_4)A_4, \tag{3}$$

where *k* is rate of interest or services of per unit of capital A_1; *l* is the wage rate per unit of labor A_2; *m* is the cost per unit of principal raw material A_3; and *n* is the cost per unit of irregular raw material A_4. In nonlinear budget equation (3) we consider (Moolio et al., 2009; Mohajan & Mohajan, 2023c),

$$n(A_4) = n_0 A_4 - n_0, \tag{4}$$

where n_0 being the discounted price of the irregular input A_4. Therefore, the nonlinear budget constraint (3) takes the form (Mohajan, 2021a; Mohajan & Mohajan, 2023d);

$$B(A_1, A_2, A_3, A_4) = kA_1 + lA_2 + mA_3 + n_0 A_4^2 - n_0 A_4. \tag{5}$$

We now formulate the maximization problem for the profit function (1) in terms of single Lagrange multiplier λ by defining the Lagrangian function $S(A_1, A_2, A_3, A_4, \lambda)$ as (Ferdous & Mohajan, 2022; Mohajan & Mohajan, 2023e),

$$S(A_1, A_2, A_3, A_4, \lambda) = \Gamma A_1^{\alpha} A_2^{\beta} A_3^{\gamma} A_4^{\delta} + \lambda\{B(A_1, A_2, A_3, A_4) - kA_1 - lA_2 - mA_3 - n_0 A_4^2 + n_0 A_4\}. \tag{6}$$

Relation (6) is a 5-dimensional unconstrained problem that is obtained from (1) and 4-dimensional constrained problem (3), where Lagrange multiplier λ, is considered as a device in our profit maximization model.

6. Analysis on Four Inputs

For maximization, first order differentiation equals to zero; then from (6) we can write (Islam et al., 2010; Mohajan, 2021b; Mohajan & Mohajan, 2022f),

$$S_\lambda = B - kA_1 - lA_2 - mA_3 - n_0 A_4^2 + n_0 A_4 = 0, \tag{7a}$$

$$S_1 = \alpha\, \Gamma A_1^{\alpha-1} A_2^{\beta} A_3^{\gamma} A_4^{\delta} - \lambda k = 0, \tag{7b}$$

$$S_2 = \beta\, \Gamma A_1^{\alpha} A_2^{\beta-1} A_3^{\gamma} A_4^{\delta} - \lambda l = 0, \tag{7c}$$

$$S_3 = \gamma\, \Gamma A_1^{\alpha} A_2^{\beta} A_3^{\gamma-1} A_4^{\delta} - \lambda m = 0, \tag{7d}$$

$$S_4 = \delta\, \Gamma A_1^{\alpha} A_2^{\beta} A_3^{\gamma} A_4^{\delta-1} - \lambda n_0 (2A_4 - 1) = 0, \tag{7e}$$

where, $\frac{\partial S}{\partial \lambda} = S_\lambda$, $\frac{\partial S}{\partial A_1} = S_1$, $\frac{\partial S}{\partial A_2} = S_2$, etc. indicate first-order partial differentiations of multivariate Lagrangian function.

Using equations (2) to (7) we can decide the values of A_1, A_2, A_3, and A_4 as follows (Ferdous & Mohajan, 2022; Mohajan, 2022; Mohajan & Mohajan, 2022b):

$$A_1 = \frac{\alpha B}{k\,\Psi}, \tag{8a}$$

$$A_2 = \frac{\beta B}{l\,\Psi}, \tag{8b}$$

$$A_3 = \frac{\gamma B}{m\,\Psi}, \tag{8c}$$

$$A_4 = \frac{\delta B}{n\,\Psi}. \tag{8d}$$

7. Bordered Hessian Matrix Analysis

Let us consider the determinant of the 5×5 bordered Hessian matrix as (Islam et al. 2010; Mohajan & Mohajan, 2023g),

$$|H| = \begin{vmatrix} 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ -B_1 & S_{11} & S_{12} & S_{13} & S_{14} \\ -B_2 & S_{21} & S_{22} & S_{23} & S_{24} \\ -B_3 & S_{31} & S_{32} & S_{33} & S_{34} \\ -B_4 & S_{41} & S_{42} & S_{43} & S_{44} \end{vmatrix}. \tag{9}$$

Taking first-order partial differentiations of (5) we get,

$$B_1 = k\,,\;\; B_2 = l\,,\;\; B_3 = m\,, \text{ and } \;\; B_4 = 2n_0 A_4 - n_0\,. \tag{10}$$

Taking second-order and cross-partial derivatives of (6) we get (Roy et al., 2021; Mohajan & Mohajan, 2023f),

$$\begin{aligned}
S_{11} &= \alpha(\alpha-1)\Gamma A_1^{\alpha-2} A_2^{\beta} A_3^{\gamma} A_4^{\delta}\,, \\
S_{22} &= \beta(\beta-1)\Gamma A_1^{\alpha} A_2^{\beta-2} A_3^{\gamma} A_4^{\delta}\,, \\
S_{33} &= \gamma(\gamma-1)\Gamma A_1^{\alpha} A_2^{\beta} A_3^{\gamma-2} A_4^{\delta}\,, \\
S_{44} &= \delta(\delta-1)\Gamma A_1^{\alpha} A_2^{\beta} A_3^{\gamma} A_4^{\delta-2}\,, \\
S_{12} &= S_{21} = \alpha\beta\,\Gamma A_1^{\alpha-1} A_2^{\beta-1} A_3^{\gamma} A_4^{\delta}\,, \\
S_{13} &= S_{31} = \alpha\gamma\,\Gamma A_1^{\alpha-1} A_2^{\beta} A_3^{\gamma-1} A_4^{\delta}\,, \\
S_{14} &= S_{41} = \alpha\delta\,\Gamma A_1^{\alpha-1} A_2^{\beta} A_3^{\gamma} A_4^{\delta-1}\,, \\
S_{23} &= S_{32} = \beta\gamma\,\Gamma A_1^{\alpha} A_2^{\beta-1} A_3^{\gamma-1} A_4^{\delta}\,, \\
S_{24} &= S_{42} = \beta\delta\,\Gamma A_1^{\alpha} A_2^{\beta-1} A_3^{\gamma} A_4^{\delta-1}\,, \\
S_{34} &= S_{43} = \gamma\delta\,\Gamma A_1^{\alpha} A_2^{\beta} A_3^{\gamma-1} A_4^{\delta-1}\,.
\end{aligned} \tag{11}$$

where $\frac{\partial^2 S}{\partial A_1 \partial A_2} = S_{12} = S_{21}$, $\frac{\partial^2 S}{\partial A_2^2} = S_{22}$, etc. indicate cross-partial, second order differentiations of multivariate Lagrangian function, respectively, etc.

Now we expand the Hessian (9) as $|H| > 0$ (Moolio et al., 2009; Mohajan et al., 2013; Mohajan & Mohajan, 2023f),

$$|H| = \frac{\Gamma^3 \alpha\beta\gamma\delta A_1^{3\alpha} A_2^{3\beta} A_3^{3\gamma} A_4^{3\delta} B^2}{A_1^2 A_2^2 A_3^2 A_4^2 \Psi^2} (\alpha + \beta + \gamma + \delta)(\delta + 3) > 0, \tag{12}$$

where efficiency parameter, $\Gamma > 0$, and budget of the firm, $B > 0$; A_1, A_2, A_3, and A_4 are four different types of inputs; and consequently, $A_1, A_2, A_3, A_4 > 0$. Parameters, $\alpha, \beta, \gamma, \delta > 0$; also in the model either $0 < \Psi = \alpha + \beta + \gamma + \delta < 1$, $\Psi = 1$ or $\Psi > 1$. Hence, equation (12) gives; $|H| > 0$ (Islam et al., 2010; Mohajan & Mohajan, 2022g, 2023d).

8. Determination of Lagrange Multiplier λ

Now using the necessary values from (8) in (7a) we get (Roy et al., 2021; Mohajan & Mohajan, 2023f),

$$B = \frac{\alpha\, \Gamma A_1^{\alpha} A_1^{\beta} A_1^{\gamma} A_1^{\delta}}{\lambda} + \frac{\beta\, \Gamma A_1^{\alpha} A_1^{\beta} A_1^{\gamma} A_1^{\delta}}{\lambda} + \frac{\gamma\, \Gamma A_1^{\alpha} A_1^{\beta} A_1^{\gamma} A_1^{\delta}}{\lambda} + \frac{\delta\, \Gamma A_1^{\alpha} A_1^{\beta} A_1^{\gamma} A_1^{\delta}}{\lambda}$$

$$\lambda = \frac{\Gamma A_1^{\alpha} A_1^{\beta} A_1^{\gamma} A_1^{\delta} \Psi}{B}. \tag{13}$$

9. Jacobian Matrix Analysis

We have observed that the second-order condition is satisfied, so that the determinant of (5) survives at the optimum, i.e., $|J| = |H|$; and hence, we can apply the implicit function theorem. Now we compute twenty-five partial derivatives, such as $\frac{\partial \lambda}{\partial k}$, $\frac{\partial A_1}{\partial k}$, $\frac{\partial A_3}{\partial l}$, $\frac{\partial A_4}{\partial B}$, etc. that are referred to as the comparative statics of the model (Chiang, 1984; Mohajan & Mohajan, 2022a).

Let $\mathbf{G}$ be the vector-valued function of ten variables $\lambda^*, A_1^*, A_2^*, A_3^*, A_4^*, k, l, m, n$, and B, and we define the function $\mathbf{G}$ for the point $(\lambda^*, A_1^*, A_2^*, A_3^*, A_4^*, k, l, m, n, B) \in R^{10}$, and take the values in R^5. By the Implicit Function Theorem of multivariable calculus, the equation (Mohajan, 2021b; Mohajan & Mohajan, 2022a, 2023c),

$$F(\lambda^*, A_1^*, A_2^*, A_3^*, A_4^*, k, l, m, n, B) = 0, \tag{14}$$

may be solved in the form of

$$\begin{bmatrix} \lambda \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \mathbf{G}(k, l, m, n, B). \tag{15}$$

Now the 5×5 Jacobian matrix for $\mathbf{G}(k, l, m, n, B)$; regarded as $J_G = \frac{\partial(\lambda, A_1, A_2, A_3, A_4)}{\partial(k, l, m, n_0, B)}$, and is represented by;

$$J_G = \begin{bmatrix} \frac{\partial\lambda}{\partial k} & \frac{\partial\lambda}{\partial l} & \frac{\partial\lambda}{\partial m} & \frac{\partial\lambda}{\partial n_0} & \frac{\partial\lambda}{\partial B} \\ \frac{\partial A_1}{\partial k} & \frac{\partial A_1}{\partial l} & \frac{\partial A_1}{\partial m} & \frac{\partial A_1}{\partial n_0} & \frac{\partial A_1}{\partial B} \\ \frac{\partial A_2}{\partial k} & \frac{\partial A_2}{\partial l} & \frac{\partial A_2}{\partial m} & \frac{\partial A_2}{\partial n_0} & \frac{\partial A_2}{\partial B} \\ \frac{\partial A_3}{\partial k} & \frac{\partial A_3}{\partial l} & \frac{\partial A_3}{\partial m} & \frac{\partial A_3}{\partial n_0} & \frac{\partial A_3}{\partial B} \\ \frac{\partial A_4}{\partial k} & \frac{\partial A_4}{\partial l} & \frac{\partial A_4}{\partial m} & \frac{\partial A_4}{\partial n_0} & \frac{\partial A_4}{\partial B} \end{bmatrix}. \tag{16}$$

$$= -J^{-1} \begin{bmatrix} -A_1 & -A_2 & -A_3 & -A_4^2 + A_4 & 1 \\ -\lambda & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & -2\lambda A_4 + \lambda & 0 \end{bmatrix}. \tag{17}$$

The inverse of Jacobian matrix is, $J^{-1} = \frac{1}{|J|} C^T$, where $C = (C_{ij})$, the matrix of cofactors of J, where T for transpose, then (17) becomes (Moolio et al., 2009; Roy et al., 2021; Mohajan, 2021c),

$$= -\frac{1}{|J|} \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} & C_{51} \\ C_{12} & C_{22} & C_{32} & C_{42} & C_{52} \\ C_{13} & C_{23} & C_{33} & C_{43} & C_{53} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{54} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} \end{bmatrix} \begin{bmatrix} -A_1 & -A_2 & -A_3 & -A_4^2 + A_4 & 1 \\ -\lambda & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & -2\lambda A_4 + \lambda & 0 \end{bmatrix}. \tag{18}$$

$$J_G = -\frac{1}{|J|} \begin{bmatrix} -A_1C_{11} - \lambda C_{21} & -A_2C_{11} - \lambda C_{31} & -A_3C_{11} - \lambda C_{41} & -A_4^2C_{11} + A_4C_{11} - 2\lambda A_4C_{51} + \lambda C_{51} & C_{11} \\ -A_1C_{12} - \lambda C_{22} & -A_2C_{12} - \lambda C_{32} & -A_3C_{12} - \lambda C_{42} & -A_4^2C_{12} + A_4C_{12} - 2\lambda A_4C_{52} + \lambda C_{52} & C_{12} \\ -A_1C_{13} - \lambda C_{23} & -A_2C_{13} - \lambda C_{33} & -A_3C_{13} - \lambda C_{43} & -A_4^2C_{13} + A_4C_{13} - 2\lambda A_4C_{53} + \lambda C_{53} & C_{13} \\ -A_1C_{14} - \lambda C_{24} & -A_2C_{14} - \lambda C_{34} & -A_3C_{14} - \lambda C_{44} & -A_4^2C_{14} + A_4C_{14} - 2\lambda A_4C_{54} + \lambda C_{54} & C_{14} \\ -A_1C_{15} - \lambda C_{25} & -A_2C_{15} - \lambda C_{35} & -A_3C_{15} - \lambda C_{45} & -A_4^2C_{15} + A_4C_{15} - 2\lambda A_4C_{55} + \lambda C_{55} & C_{15} \end{bmatrix}. \tag{19}$$

In (19) there are total 25 comparative statics, and in this study, we shall deal only with five of them. We shall study the economic analysis of Lagrange multiplier when per unit costs of various inputs are increased. Now we consider that the firm always attempts for the profit maximization production (Baxley & Moorhouse, 1984; Islam et al., 2010).

10. Sensitivity Analysis

Now we analyze the economic effects on Lagrange multiplier λ when budget of the industry increases. Taking T_{15} (i.e., term of 1st row and 5th column) from both sides of (19) we get (Moolio et al., 2009; Islam et al., 2011; Roy et al., 2021),

$$\frac{\partial \lambda}{\partial B} = -\frac{1}{J}\left[C_{11}\right]$$

$$= -\frac{1}{|J|}\text{Cofactor of } C_{11}$$

$$= -\frac{1}{|J|}\begin{vmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{vmatrix}$$

$$= -\frac{1}{|J|}\left\{ S_{11}\begin{vmatrix} S_{22} & S_{23} & S_{24} \\ S_{32} & S_{33} & S_{34} \\ S_{42} & S_{43} & S_{44} \end{vmatrix} - S_{12}\begin{vmatrix} S_{21} & S_{23} & S_{24} \\ S_{31} & S_{33} & S_{34} \\ S_{41} & S_{43} & S_{44} \end{vmatrix} + S_{13}\begin{vmatrix} S_{21} & S_{22} & S_{24} \\ S_{31} & S_{32} & S_{34} \\ S_{41} & S_{42} & S_{44} \end{vmatrix} - S_{14}\begin{vmatrix} S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \\ S_{41} & S_{42} & S_{43} \end{vmatrix} \right\}$$

$$= \frac{A_1}{|J|}\left[-S_{11}\left\{S_{22}\left(S_{33}S_{44} - S_{43}S_{34}\right) + S_{23}\left(S_{42}S_{34} - S_{32}S_{44}\right) + S_{24}\left(S_{32}S_{43} - S_{42}S_{33}\right)\right\}\right.$$

$$-S_{12}\left\{-S_{21}\left(S_{33}S_{44} - S_{43}S_{34}\right) + S_{23}\left(-S_{41}S_{34} + S_{31}S_{44}\right) + S_{24}\left(-S_{31}S_{43} + S_{41}S_{33}\right)\right\}$$

$$+S_{13}\left\{-S_{21}\left(S_{32}S_{44} - S_{42}S_{34}\right) + S_{22}\left(-S_{41}S_{34} + S_{31}S_{44}\right) + S_{24}\left(-S_{31}S_{42} + S_{41}S_{32}\right)\right\}$$

$$\left.-S_{14}\left\{-S_{21}\left(S_{32}S_{43} - S_{42}S_{33}\right) + S_{22}\left(-S_{41}S_{33} + S_{31}S_{43}\right) + S_{23}\left(-S_{31}S_{42} + S_{41}S_{32}\right)\right\}\right]$$

$$= \frac{1}{|J|}\{ -S_{11}S_{22}S_{33}S_{44} + S_{11}S_{24}S_{42}S_{33} - S_{11}S_{24}S_{32}S_{43} + S_{11}S_{23}S_{32}S_{44} - S_{11}S_{23}S_{42}S_{34} + S_{11}S_{22}S_{43}S_{34}$$

$$+ S_{12}S_{21}S_{33}S_{44} - S_{12}S_{21}S_{43}S_{34} + S_{12}S_{23}S_{41}S_{34} - S_{12}S_{23}S_{31}S_{44} + S_{12}S_{24}S_{31}S_{43} - S_{12}S_{24}S_{41}S_{33}$$

$$- S_{13}S_{21}S_{32}S_{44} + S_{13}S_{21}S_{42}S_{34} - S_{13}S_{22}S_{41}S_{34} + S_{13}S_{22}S_{31}S_{44} - S_{13}S_{24}S_{31}S_{42} + S_{13}S_{24}S_{41}S_{32}$$

$$+ S_{14}S_{21}S_{32}S_{43} - S_{14}S_{21}S_{42}S_{33} + S_{14}S_{22}S_{41}S_{33} - S_{14}S_{22}S_{31}S_{43} + S_{14}S_{23}S_{31}S_{42} - S_{14}S_{23}S_{41}S_{32} \}$$

$$= \frac{1}{|J|}\frac{\Gamma^4 A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2 A_2^2 A_3^2 A_4^2}\{ -\alpha(\alpha-1)\beta(\beta-1)\gamma(\gamma-1)\delta(\delta-1) + \alpha(\alpha-1)\beta^2\gamma(\gamma-1)\delta^2$$

$$-\alpha(\alpha-1)\beta^2\gamma^2\delta^2 + \alpha(\alpha-1)\beta^2\gamma^2\delta(\delta-1) - \alpha(\alpha-1)\beta^2\gamma^2\delta^2 + \alpha(\alpha-1)\beta(\beta-1)\gamma^2\delta^2$$

$$+\alpha^2\beta^2\gamma(\gamma-1)\delta(\delta-1) - \alpha^2\beta^2\gamma^2\delta^2 + \alpha^2\beta^2\gamma^2\delta^2 - \alpha^2\beta^2\gamma^2\delta(\delta-1) + \alpha^2\beta^2\gamma^2\delta^2$$

$$-\alpha^2\beta^2\gamma(\gamma-1)\delta^2 \quad -\alpha^2\beta^2\gamma^2\delta(\delta-1) \quad +\alpha^2\beta^2\gamma^2\delta^2 \quad -\alpha^2\beta(\beta-1)\gamma^2\delta^2 \quad +\alpha^2\beta(\beta-1)\gamma^2\delta(\delta-1)$$

$$-\alpha^2\beta^2\gamma^2\delta^2 \quad +\alpha^2\beta^2\gamma^2\delta^2 \quad +\alpha^2\beta^2\gamma^2\delta^2 \quad -\alpha^2\beta^2\gamma(\gamma-1)\delta^2 \quad -\alpha^2\beta(\beta-1)\gamma^2\delta^2 \quad +\alpha^2\beta^2\gamma^2\delta^2$$

$$-\alpha^2\beta^2\gamma^2\delta^2\}$$

$$=\frac{1}{|J|}\frac{\Gamma^4\alpha\beta\gamma\delta A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2^2A_3^2A_4^2}\{-(\alpha-1)(\beta-1)(\gamma-1)(\delta-1) \quad +(\alpha-1)\beta\gamma(\delta-1) \quad +\alpha\beta(\gamma-1)(\delta-1)$$

$$-2\alpha\beta\gamma(\delta-1) \quad +\alpha(\beta-1)\gamma(\delta-1) \quad -2\alpha\beta(\gamma-1)\delta \quad +3\alpha\beta\gamma\delta \quad -2(\alpha-1)\beta\gamma\delta \quad +(\alpha-1)(\beta-1)\gamma\delta$$

$$+(\alpha-1)\beta(\gamma-1)\delta-3\alpha(\beta-1)\gamma\delta\}$$

$$\frac{\partial\lambda}{\partial B}=\frac{1}{|J|}\frac{\Gamma^4\alpha\beta\gamma\delta A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2^2A_3^2A_4^2}\left(-2\alpha\beta\gamma\delta+2\alpha\gamma\delta+\alpha\beta\delta+\beta\delta+\alpha+\beta+\gamma-1\right). \tag{20}$$

Now we consider $\alpha=\beta=\gamma=\delta=\frac{1}{2}$ then we get, $\Psi=2$, i.e., for increasing returns to scale, in (20) we get,

$$\frac{\partial\lambda}{\partial B}=\frac{\Gamma^4}{2^4|J|}>0. \tag{21}$$

From the relation (21) we see that when budget of the industry increases, the level of Lagrange multiplier, i.e., marginal profit also increases, which is reasonable. Hence, increasing returns to scale is suitable for the industry. In this circumstance profit maximization attempts may be successful, and industry may be sustainable.

Now we consider $\alpha=\beta=\gamma=\delta=\frac{1}{4}$ then we get, $\Psi=1$, i.e., for constant returns to scale, in (20) we get,

$$\frac{\partial\lambda}{\partial B}=-\frac{19\Gamma^4}{2^{13}A_1A_2A_3A_4|J|}<0. \tag{22}$$

From the relation (22) we see that when budget of the industry increases, the level of Lagrange multiplier, i.e., marginal profit decreases. Consequently, the industry faces unsustainable atmosphere. Hence, in this situation constant return to scale is not suitable for the industry.

Now we consider $\alpha=\beta=\gamma=\frac{1}{8}$ and $\delta=\frac{1}{2}$ then we get, $\Psi=\frac{7}{8}$, i.e., for decreasing returns to scale, in (20) we get,

$$\frac{\partial\lambda}{\partial B}=-\frac{277\Gamma^4}{2^7|J|}<0. \tag{23}$$

From the relation (23) we see that it provides same property as in (22). Hence, both constant and decreasing returns to scale are not suitable for the sustainable environment of the industry when budget of the industry increases.

Now we analyze the economic effects on Lagrange multiplier λ when interest rate of capital increases. Taking T_{11} (i.e., term of 1st row and 1st column) from both sides of (19) we get (Islam et al., 2011; Roy et al., 2021),

$$\frac{\partial \lambda}{\partial k}=\frac{A_1}{J}[C_{11}]+\frac{\lambda}{J}[C_{21}]$$

$$=\frac{A_1}{|J|}\text{Cofactor of } C_{11}+\frac{\lambda}{|J|}\text{Cofactor of } \mathrm{C}_{21}$$

$$=\frac{A_1}{|J|}\begin{vmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{vmatrix}-\frac{\lambda}{|J|}\begin{vmatrix} -B_1 & -B_2 & -B_3 & -B_4 \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{vmatrix}$$

$$=\frac{A_1}{|J|}\left\{S_{11}\begin{vmatrix} S_{22} & S_{23} & S_{24} \\ S_{32} & S_{33} & S_{34} \\ S_{42} & S_{43} & S_{44} \end{vmatrix}-S_{12}\begin{vmatrix} S_{21} & S_{23} & S_{24} \\ S_{31} & S_{33} & S_{34} \\ S_{41} & S_{43} & S_{44} \end{vmatrix}+S_{13}\begin{vmatrix} S_{21} & S_{22} & S_{24} \\ S_{31} & S_{32} & S_{34} \\ S_{41} & S_{42} & S_{44} \end{vmatrix}-S_{14}\begin{vmatrix} S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \\ S_{41} & S_{42} & S_{43} \end{vmatrix}\right\}$$

$$-\frac{\lambda}{|J|}\left\{-B_1\begin{vmatrix} S_{22} & S_{23} & S_{24} \\ S_{32} & S_{33} & S_{34} \\ S_{42} & S_{43} & S_{44} \end{vmatrix}+B_2\begin{vmatrix} S_{21} & S_{23} & S_{24} \\ S_{31} & S_{33} & S_{34} \\ S_{41} & S_{43} & S_{44} \end{vmatrix}-B_3\begin{vmatrix} S_{21} & S_{22} & S_{24} \\ S_{31} & S_{32} & S_{34} \\ S_{41} & S_{42} & S_{44} \end{vmatrix}+B_4\begin{vmatrix} S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \\ S_{41} & S_{42} & S_{43} \end{vmatrix}\right\}$$

$$=\frac{A_1}{|J|}\left[-S_{11}\{S_{22}(S_{33}S_{44}-S_{43}S_{34})+S_{23}(S_{42}S_{34}-S_{32}S_{44})+S_{24}(S_{32}S_{43}-S_{42}S_{33})\}\right.$$

$$-S_{12}\{-S_{21}(S_{33}S_{44}-S_{43}S_{34})+S_{23}(-S_{41}S_{34}+S_{31}S_{44})+S_{24}(-S_{31}S_{43}+S_{41}S_{33})\}$$

$$+S_{13}\{-S_{21}(S_{32}S_{44}-S_{42}S_{34})+S_{22}(-S_{41}S_{34}+S_{31}S_{44})+S_{24}(-S_{31}S_{42}+S_{41}S_{32})\}$$

$$\left.-S_{14}\{-S_{21}(S_{32}S_{43}-S_{42}S_{33})+S_{22}(-S_{41}S_{33}+S_{31}S_{43})+S_{23}(-S_{31}S_{42}+S_{41}S_{32})\}\right]$$

$$-\frac{\lambda}{|J|}\left[-B_1\{S_{22}(S_{33}S_{44}-S_{43}S_{34})+S_{23}(S_{42}S_{34}-S_{32}S_{44})+S_{24}(S_{32}S_{43}-S_{42}S_{33})\}\right.$$

$$+B_2\{-S_{21}(S_{33}S_{44}-S_{43}S_{34})+S_{23}(-S_{41}S_{34}+S_{31}S_{44})+S_{24}(-S_{31}S_{43}+S_{41}S_{33})\}$$

$$-B_3\{-S_{21}(S_{32}S_{44}-S_{42}S_{34})+S_{22}(-S_{41}S_{34}+S_{31}S_{44})+S_{24}(-S_{31}S_{42}+S_{41}S_{32})\}$$

$$\left.+B_4\{-S_{21}(S_{32}S_{43}-S_{42}S_{33})+S_{22}(-S_{41}S_{33}+S_{31}S_{43})+S_{23}(-S_{31}S_{42}+S_{41}S_{32})\}\right]$$

$$=\frac{A_1}{|J|}\{-S_{11}S_{22}S_{33}S_{44}\quad+S_{11}S_{24}S_{42}S_{33}\quad-S_{11}S_{24}S_{32}S_{43}\quad+S_{11}S_{23}S_{32}S_{44}\quad-S_{11}S_{23}S_{42}S_{34}\quad+S_{11}S_{22}S_{43}S_{34}$$

$$+S_{12}S_{21}S_{33}S_{44}\quad-S_{12}S_{21}S_{43}S_{34}\quad+S_{12}S_{23}S_{41}S_{34}\quad-S_{12}S_{23}S_{31}S_{44}\quad+S_{12}S_{24}S_{31}S_{43}\quad-S_{12}S_{24}S_{41}S_{33}$$

$$-S_{13}S_{21}S_{32}S_{44} \quad +S_{13}S_{21}S_{42}S_{34} \quad -S_{13}S_{22}S_{41}S_{34} \quad +S_{13}S_{22}S_{31}S_{44} \quad -S_{13}S_{24}S_{31}S_{42} \quad +S_{13}S_{24}S_{41}S_{32}$$

$$+S_{14}S_{21}S_{32}S_{43} \quad -S_{14}S_{21}S_{42}S_{33} \quad +S_{14}S_{22}S_{41}S_{33} \quad -S_{14}S_{22}S_{31}S_{43} \quad +S_{14}S_{23}S_{31}S_{42} \quad -S_{14}S_{23}S_{41}S_{32}\}$$

$$-\frac{\lambda}{|J|}\{-B_1S_{22}S_{33}S_{44} \quad +B_1S_{22}S_{43}S_{34} \quad -B_1S_{23}S_{42}S_{34} \quad +B_1S_{23}S_{32}S_{44} \quad -B_1S_{24}S_{32}S_{43} \quad +B_1S_{24}S_{42}S_{33}$$

$$-B_2S_{21}S_{33}S_{44} \quad +B_2S_{21}S_{43}S_{34} \quad -B_2S_{23}S_{41}S_{34} \quad +B_2S_{23}S_{31}S_{44} \quad -B_2S_{24}S_{31}S_{43} \quad +B_2S_{24}S_{41}S_{33}$$

$$+B_3S_{21}S_{32}S_{44} \quad -B_3S_{21}S_{42}S_{34} \quad +B_3S_{22}S_{41}S_{34} \quad -B_3S_{22}S_{31}S_{44} \quad +B_3S_{24}S_{31}S_{42} \quad -B_3S_{24}S_{41}S_{32}$$

$$-B_4S_{21}S_{32}S_{43}+B_4S_{21}S_{42}S_{33}-B_4S_{22}S_{41}S_{33}+B_4S_{22}S_{31}S_{43}-B_4S_{23}S_{31}S_{42}+B_4S_{23}S_{41}S_{32}\}$$

$$=\frac{A_1}{|J|}\frac{\Gamma^4A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2^2A_3^2A_4^2}\{-\alpha(\alpha-1)\beta(\beta-1)\gamma(\gamma-1)\delta(\delta-1) \quad +\alpha(\alpha-1)\beta^2\gamma(\gamma-1)\delta^2$$

$$-\alpha(\alpha-1)\beta^2\gamma^2\delta^2 \quad +\alpha(\alpha-1)\beta^2\gamma^2\delta(\delta-1) \quad -\alpha(\alpha-1)\beta^2\gamma^2\delta^2 \quad +\alpha(\alpha-1)\beta(\beta-1)\gamma^2\delta^2$$

$$+\alpha^2\beta^2\gamma(\gamma-1)\delta(\delta-1) \quad -\alpha^2\beta^2\gamma^2\delta^2 \quad +\alpha^2\beta^2\gamma^2\delta^2 \quad -\alpha^2\beta^2\gamma^2\delta(\delta-1) \quad +\alpha^2\beta^2\gamma^2\delta^2$$

$$-\alpha^2\beta^2\gamma(\gamma-1)\delta^2 \quad -\alpha^2\beta^2\gamma^2\delta(\delta-1) \quad +\alpha^2\beta^2\gamma^2\delta^2 \quad -\alpha^2\beta(\beta-1)\gamma^2\delta^2 \quad +\alpha^2\beta(\beta-1)\gamma^2\delta(\delta-1)$$

$$-\alpha^2\beta^2\gamma^2\delta^2 \quad +\alpha^2\beta^2\gamma^2\delta^2 \quad +\alpha^2\beta^2\gamma^2\delta^2 \quad -\alpha^2\beta^2\gamma(\gamma-1)\delta^2 \quad -\alpha^2\beta(\beta-1)\gamma^2\delta^2 \quad +\alpha^2\beta^2\gamma^2\delta^2$$

$$-\alpha^2\beta^2\gamma^2\delta^2\} \qquad -\frac{\Gamma^3A_1^{3\alpha}A_2^{3\beta}A_3^{3\gamma}A_4^{3\delta}}{|J|A_1^2A_2^2A_3^2A_4^2}\frac{\Gamma A_1^{\alpha}A_1^{\beta}A_1^{\gamma}A_1^{\delta}\Psi}{B} \qquad \{-kA_1^2\beta(\beta-1)\gamma(\gamma-1)\delta(\delta-1)$$

$$+kA_1^2\beta(\beta-1)\gamma^2\delta^2 \quad -kA_1^2\beta^2\gamma^2\delta^2 \quad +kA_1^2\beta^2\gamma^2\delta(\delta-1) \quad -kA_1^2\beta^2\gamma^2\delta^2 \quad +kA_1^2\beta^2\gamma(\gamma-1)\delta^2$$

$$-lA_1A_2\alpha\beta\gamma(\gamma-1)\delta(\delta-1) \quad +lA_1A_2\alpha\beta\gamma^2\delta^2 \quad -lA_1A_2\alpha\beta\gamma^2\delta^2 \quad +lA_1A_2\alpha\beta\gamma^2\delta(\delta-1) \quad -lA_1A_2\alpha\beta\gamma^2\delta^2$$

$$+lA_2A_4\alpha\beta\gamma(\gamma-1)\delta^2 \quad +mA_1A_3\alpha\beta^2\gamma\delta(\delta-1) \quad -mA_1A_3\alpha\beta^2\gamma\delta^2 \quad +mA_1A_3\alpha\beta(\beta-1)\gamma\delta^2$$

$$-mA_1A_3\alpha\beta(\beta-1)\gamma\delta(\delta-1) \quad +mA_1A_3\alpha\beta^2\gamma\delta^2 \quad -mA_1A_3\alpha\beta^2\gamma\delta^2 \quad -nA_1A_4\alpha\beta^2\gamma^2\delta$$

$$+nA_1A_4\alpha\beta^2\gamma(\gamma-1)\delta \quad -nA_1A_4\alpha\beta(\beta-1)\gamma(\gamma-1)\delta \quad +nA_1A_4\alpha\beta(\beta-1)\gamma^2\delta \quad -nA_1A_4\alpha\beta^2\gamma^2\delta$$

$$+nA_1A_4\alpha\beta^2\gamma^2\delta\}$$

$$=\frac{1}{|J|}\frac{\Gamma^4\alpha\beta\gamma\delta A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1A_2^2A_3^2A_4^2}\{-(\alpha-1)(\beta-1)(\gamma-1)(\delta-1) \quad +(\alpha-1)\beta\gamma(\delta-1) \quad +\alpha\beta(\gamma-1)(\delta-1)$$

$$-2\alpha\beta\gamma(\delta-1) \quad +\alpha(\beta-1)\gamma(\delta-1) \quad -2\alpha\beta(\gamma-1)\delta \quad +3\alpha\beta\gamma\delta \quad -2(\alpha-1)\beta\gamma\delta \quad +(\alpha-1)(\beta-1)\gamma\delta$$

$$+(\alpha-1)\beta(\gamma-1)\delta \quad -3\alpha(\beta-1)\gamma\delta\} \quad -\frac{1}{|J|}\frac{\Gamma^4\alpha\beta\gamma\delta A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1A_2^2A_3^2A_4^2} \quad \{-(\beta-1)(\gamma-1)(\delta-1)$$

$$+3\beta\gamma(\delta-1)-\beta(\gamma-1)(\delta-1)-(\beta-1)\gamma(\delta-1)+2(\beta-1)\gamma\delta-4\beta\gamma\delta+2\beta(\gamma-1)\delta-(2A_4-1)\beta\gamma\delta$$

$$+(2A_4-1)(\beta-1)\gamma\delta+(2A_4-1)\beta(\gamma-1)\delta-(2A_4-1)(\beta-1)(\gamma-1)\delta\}$$

$$\frac{\partial\lambda}{\partial k}=\frac{1}{|J|}\frac{\Gamma^4\alpha\beta\gamma\delta A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1A_2^2A_3^2A_4^2}\left(2A_4\delta-3\alpha\beta\gamma\delta+2\alpha\gamma\delta+2\alpha\beta\delta+\beta\delta-\alpha\delta+\alpha+3\beta+3\gamma+\delta-2\right). \quad (24)$$

Now we consider $\alpha=\beta=\gamma=\delta=\frac{1}{2}$ then we get, $\Psi=2$, i.e., for increasing returns to scale, in (24) we get,

$$\frac{\partial\lambda}{\partial k}=\frac{\Gamma^4A_1}{2^8|J|}\left(16A_4+37\right)>0. \quad (25)$$

From the relation (25) we see that when interest rate of capital increases, the level of Lagrange multiplier, i.e., marginal profit also increases. We believe that for increasing returns to scale profit maximization is possible for this industry. Therefore, in this situation we think that the industry is in sustainable position, even when interest rate of capital increases.

Now we consider $\alpha=\beta=\gamma=\delta=\frac{1}{4}$ then we get, $\Psi=1$, i.e., for constant returns to scale, in (24) we get,

$$\frac{\partial\lambda}{\partial k}=\frac{\Gamma^4}{2^{16}A_2A_3A_4|J|}\left(128A_4+13\right)>0. \quad (26)$$

From the relation (26) we face the same case as in (25). Therefore, both increasing and constant returns to scale are suitable for the industry, and it can achieve profit maximization environment.

Now we consider $\alpha=\beta=\gamma=\frac{1}{8}$ and $\delta=\frac{1}{2}$ then we get, $\Psi=\frac{7}{8}$, i.e., for decreasing returns to scale, in (24) we get,

$$\frac{\partial\lambda}{\partial k}=\frac{\Gamma^4}{2^{20}A_1^{\frac{1}{2}}A_2^{\frac{3}{2}}A_3^{\frac{3}{2}}|J|}\left(1024A_4-611\right). \quad (27)$$

In (27) if $A_4>\frac{611}{1024}$ we get,

$$\frac{\partial\lambda}{\partial k}>0. \quad (28)$$

From the inequality (28) we see that when interest rate of capital increases, the level of Lagrange multiplier, i.e., marginal profit also increases. In this situation the industry can move to profit maximization production procedure.

In (27) if $A_4<\frac{611}{1024}$ we get,

$$\frac{\partial\lambda}{\partial k}<0. \quad (29)$$

From the inequality (29) we see that when interest rate of capital increases, the level of Lagrange multiplier, i.e., marginal profit decreases. In this situation the industry may face unsustainable atmosphere.

In (27) if $A_4 = \frac{611}{1024}$ we get,

$$\frac{\partial \lambda}{\partial k} = 0 . \tag{30}$$

From the inequality (30) we see that when interest rate of capital increases, there is no change of the level of Lagrange multiplier. It seems that in this circumstance there is no effect on Lagrange multiplier when interest rate of capital is increased or decreased.

Now we analyze the economic effects on Lagrange multiplier λ when wage rate of the workers increases. Taking T_{12} (i.e., term of 1st row and 2nd column) from both sides of (19) we get (Moolio et al., 2009; Roy et al., 2021, Mohajan, 2022),

$$\frac{\partial \lambda}{\partial l} = \frac{A_2}{J}[C_{11}] + \frac{\lambda}{J}[C_{31}]$$

$$= \frac{A_2}{|J|}\text{Cofactor of } C_{11} + \frac{\lambda}{|J|}\text{Cofactor of } \mathrm{C}_{31}$$

$$= \frac{A_2}{|J|}\begin{vmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{vmatrix} + \frac{\lambda}{|J|}\begin{vmatrix} -B_1 & -B_2 & -B_3 & -B_4 \\ S_{11} & S_{12} & S_{13} & S_{14} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{vmatrix}$$

$$= \frac{A_2}{|J|}\left\{ S_{11}\begin{vmatrix} S_{22} & S_{23} & S_{24} \\ S_{32} & S_{33} & S_{34} \\ S_{42} & S_{43} & S_{44} \end{vmatrix} - S_{12}\begin{vmatrix} S_{21} & S_{23} & S_{24} \\ S_{31} & S_{33} & S_{34} \\ S_{41} & S_{43} & S_{44} \end{vmatrix} + S_{13}\begin{vmatrix} S_{21} & S_{22} & S_{24} \\ S_{31} & S_{32} & S_{34} \\ S_{41} & S_{42} & S_{44} \end{vmatrix} - S_{14}\begin{vmatrix} S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \\ S_{41} & S_{42} & S_{43} \end{vmatrix} \right\}$$

$$+ \frac{\lambda}{|J|}\left\{ -B_1\begin{vmatrix} S_{12} & S_{13} & S_{14} \\ S_{32} & S_{33} & S_{34} \\ S_{42} & S_{43} & S_{44} \end{vmatrix} + B_2\begin{vmatrix} S_{11} & S_{13} & S_{14} \\ S_{31} & S_{33} & S_{34} \\ S_{41} & S_{43} & S_{44} \end{vmatrix} - B_3\begin{vmatrix} S_{11} & S_{12} & S_{14} \\ S_{31} & S_{32} & S_{34} \\ S_{41} & S_{42} & S_{44} \end{vmatrix} + B_4\begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{31} & S_{32} & S_{33} \\ S_{41} & S_{42} & S_{43} \end{vmatrix} \right\}$$

$$= \frac{A_2}{|J|}\left[-S_{11}\left\{ S_{22}\left(S_{33}S_{44} - S_{43}S_{34}\right) + S_{23}\left(S_{42}S_{34} - S_{32}S_{44}\right) + S_{24}\left(S_{32}S_{43} - S_{42}S_{33}\right) \right\} \right.$$

$$- S_{12}\left\{ -S_{21}\left(S_{33}S_{44} - S_{43}S_{34}\right) + S_{23}\left(-S_{41}S_{34} + S_{31}S_{44}\right) + S_{24}\left(-S_{31}S_{43} + S_{41}S_{33}\right) \right\}$$

$$+ S_{13}\left\{ -S_{21}\left(S_{32}S_{44} - S_{42}S_{34}\right) + S_{22}\left(-S_{41}S_{34} + S_{31}S_{44}\right) + S_{24}\left(-S_{31}S_{42} + S_{41}S_{32}\right) \right\}$$

$$\left. - S_{14}\left\{ -S_{21}\left(S_{32}S_{43} - S_{42}S_{33}\right) + S_{22}\left(-S_{41}S_{33} + S_{31}S_{43}\right) + S_{23}\left(-S_{31}S_{42} + S_{41}S_{32}\right) \right\} \right]$$

$$+ \frac{\lambda}{|J|}\left[-B_1\left\{ S_{12}\left(S_{33}S_{44} - S_{43}S_{34}\right) + S_{13}\left(S_{42}S_{34} - S_{32}S_{44}\right) + S_{14}\left(S_{32}S_{43} - S_{42}S_{33}\right) \right\} \right.$$

$$+ B_2\{-S_{11}(S_{33}S_{44} - S_{43}S_{34}) + S_{13}(S_{41}S_{34} - S_{31}S_{44}) + S_{14}(S_{31}S_{43} - S_{41}S_{33})\}$$

$$- B_3\{S_{11}(S_{32}S_{44} - S_{42}S_{34}) + S_{12}(S_{41}S_{34} - S_{31}S_{44}) + S_{13}(S_{31}S_{42} - S_{41}S_{32})\}$$

$$+ B_4\{S_{11}(S_{32}S_{43} - S_{42}S_{33}) + S_{12}(S_{41}S_{33} - S_{31}S_{43}) + S_{13}(S_{31}S_{42} - S_{41}S_{32})\}]$$

$$= \frac{A_2}{|J|}\{-S_{11}S_{22}S_{33}S_{44} + S_{11}S_{24}S_{42}S_{33} - S_{11}S_{24}S_{32}S_{43} + S_{11}S_{23}S_{32}S_{44} - S_{11}S_{23}S_{42}S_{34} + S_{11}S_{22}S_{43}S_{34}$$

$$+ S_{12}S_{21}S_{33}S_{44} - S_{12}S_{21}S_{43}S_{34} + S_{12}S_{23}S_{41}S_{34} - S_{12}S_{23}S_{31}S_{44} + S_{12}S_{24}S_{31}S_{43} - S_{12}S_{24}S_{41}S_{33}$$

$$- S_{13}S_{21}S_{32}S_{44} + S_{13}S_{21}S_{42}S_{34} - S_{13}S_{22}S_{41}S_{34} + S_{13}S_{22}S_{31}S_{44} - S_{13}S_{24}S_{31}S_{42} + S_{13}S_{24}S_{41}S_{32}$$

$$+ S_{14}S_{21}S_{32}S_{43} - S_{14}S_{21}S_{42}S_{33} + S_{14}S_{22}S_{41}S_{33} - S_{14}S_{22}S_{31}S_{43} + S_{14}S_{23}S_{31}S_{42} - S_{14}S_{23}S_{41}S_{32}\}$$

$$+ \frac{\lambda}{|J|}\{-B_1S_{12}S_{33}S_{44} + B_1S_{12}S_{43}S_{34} - B_1S_{13}S_{42}S_{34} + B_1S_{13}S_{32}S_{44} - B_1S_{14}S_{32}S_{43} + B_1S_{14}S_{42}S_{33}$$

$$- B_2S_{11}S_{33}S_{44} + B_2S_{11}S_{43}S_{34} + B_2S_{13}S_{41}S_{34} - B_2S_{13}S_{31}S_{44} + B_2S_{14}S_{31}S_{43} - B_2S_{14}S_{41}S_{33}$$

$$- B_3S_{11}S_{32}S_{44} + B_3S_{11}S_{42}S_{34} - B_3S_{12}S_{41}S_{34} + B_3S_{12}S_{31}S_{44}$$

$$- B_3S_{13}S_{31}S_{42} + B_3S_{13}S_{41}S_{32} + B_4S_{11}S_{32}S_{43} - B_4S_{11}S_{42}S_{33} + B_4S_{12}S_{41}S_{33} - B_4S_{12}S_{31}S_{43} + B_4S_{13}S_{31}S_{42}$$

$$- B_4S_{13}S_{41}S_{32}\}$$

$$= \frac{A_2}{|J|}\frac{\Gamma^4 A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2^2A_3^2A_4^2}\{-\alpha(\alpha-1)\beta(\beta-1)\gamma(\gamma-1)\delta(\delta-1) + \alpha(\alpha-1)\beta^2\gamma(\gamma-1)\delta^2$$

$$-\alpha(\alpha-1)\beta^2\gamma^2\delta^2 + \alpha(\alpha-1)\beta^2\gamma^2\delta(\delta-1) - \alpha(\alpha-1)\beta^2\gamma^2\delta^2 + \alpha(\alpha-1)\beta(\beta-1)\gamma^2\delta^2$$

$$+\alpha^2\beta^2\gamma(\gamma-1)\delta(\delta-1) - \alpha^2\beta^2\gamma^2\delta^2 + \alpha^2\beta^2\gamma^2\delta^2 - \alpha^2\beta^2\gamma^2\delta(\delta-1) + \alpha^2\beta^2\gamma^2\delta^2$$

$$-\alpha^2\beta^2\gamma(\gamma-1)\delta^2 - \alpha^2\beta^2\gamma^2\delta(\delta-1) + \alpha^2\beta^2\gamma^2\delta^2 - \alpha^2\beta(\beta-1)\gamma^2\delta^2 + \alpha^2\beta(\beta-1)\gamma^2\delta(\delta-1)$$

$$-\alpha^2\beta^2\gamma^2\delta^2 + \alpha^2\beta^2\gamma^2\delta^2 + \alpha^2\beta^2\gamma^2\delta^2 - \alpha^2\beta^2\gamma(\gamma-1)\delta^2 - \alpha^2\beta(\beta-1)\gamma^2\delta^2 + \alpha^2\beta^2\gamma^2\delta^2$$

$$-\alpha^2\beta^2\gamma^2\delta^2\} + \frac{1}{|J|}\frac{\Gamma^3 A_1^{3\alpha}A_2^{3\beta}A_3^{3\gamma}A_4^{3\delta}}{A_1^2A_2^2A_3^2A_4^2}\frac{\Gamma A_1^{\alpha}A_1^{\beta}A_1^{\gamma}A_1^{\delta}\Psi}{B}\{-kA_1A_2\alpha\beta\gamma(\gamma-1)\delta(\delta-1)$$

$$+ kA_1A_2\alpha\beta\gamma(\gamma-1)\delta(\delta-1) - kA_1A_2\alpha\beta\gamma^2\delta^2 + kA_1A_2\alpha\beta\gamma^2\delta(\delta-1) - kA_1A_2\alpha\beta\gamma^2\delta^2$$

$$+ kA_1A_2\alpha\beta\gamma(\gamma-1)\delta^2 - lA_1A_2\alpha(\alpha-1)\gamma(\gamma-1)\delta(\delta-1)$$

$$+lA_2^2\alpha(\alpha-1)\gamma^2\delta^2+lA_2^2\alpha^2\gamma^2\delta^2-lA_2^2\alpha^2\gamma^2\delta(\delta-1)+lA_2^2\alpha^2\gamma^2\delta^2-lA_2^2\alpha^2\gamma(\gamma-1)\delta^2$$

$$-mA_2A_3\alpha(\alpha-1)\beta\gamma\delta(\delta-1)\quad+mA_2A_3\alpha(\alpha-1)\beta\gamma\delta^2\quad-mA_2A_3\alpha^2\beta\gamma\delta^2\quad+mA_2A_3\alpha^2\beta\gamma\delta(\delta-1)$$

$$-mA_2A_3\alpha^2\beta\gamma^2\delta+mA_2A_3\alpha^2\beta\gamma^2\delta+nA_2A_4\alpha(\alpha-1)\beta\gamma^2\delta-nA_2A_4\alpha(\alpha-1)\beta\gamma(\gamma-1)\delta$$

$$+nA_2A_4\alpha^2\beta\gamma(\gamma-1)\delta-nA_2A_4\alpha^2\beta\gamma^2\delta+nA_2A_4\alpha^2\beta\gamma^2\delta-nA_2A_4\alpha^2\beta\gamma^2\delta\}$$

$$=\frac{1}{|J|}\frac{\Gamma^4\alpha\beta\gamma\delta A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2A_3^2A_4^2}\{-(\alpha-1)(\beta-1)(\gamma-1)(\delta-1)\quad+(\alpha-1)\beta\gamma(\delta-1)\quad+\alpha\beta(\gamma-1)(\delta-1)$$

$$-2\alpha\beta\gamma(\delta-1)\quad+\alpha(\beta-1)\gamma(\delta-1)\quad+(\alpha-1)\beta(\gamma-1)\delta\quad-2\alpha\beta(\gamma-1)\delta\quad+3\alpha\beta\gamma\delta\quad-2(\alpha-1)\beta\gamma\delta$$

$$+(\alpha-1)(\beta-1)\gamma\delta\qquad-3\alpha(\beta-1)\gamma\delta\}\qquad+\frac{1}{|J|}\frac{\Gamma^4\alpha\beta\gamma\delta A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2A_3^2A_4^2}\qquad\{-(\alpha-1)(\gamma-1)(\delta-1)$$

$$-(\alpha-1)\gamma(\delta-1)\;+\alpha\gamma(\delta-1)\;+2(\alpha-1)\gamma\delta\;-\alpha\gamma\delta\;+(2A_4-1)(\alpha-1)\gamma\delta\;-(2A_4-1)(\alpha-1)(\gamma-1)\delta$$

$$+(2A_4-1)\alpha(\gamma-1)\delta\;-(2A_4-1)\alpha\gamma\delta\}$$

$$\frac{\partial\lambda}{\partial l}=\frac{1}{|J|}\frac{\Gamma^4\alpha\beta\gamma\delta A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2A_3^2A_4^2}\left(-2A_4\delta+\alpha\beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta-2\gamma\delta-\beta\delta-\alpha\gamma+\beta+\gamma\right).\tag{31}$$

Now we consider $\alpha=\beta=\gamma=\delta=\frac{1}{2}$ then we get, $\Psi=2$, i.e., for increasing returns to scale, in (31) we get,

$$\frac{\partial\lambda}{\partial l}=\frac{1}{|J|}\frac{\Gamma^4A_2(3-16A_4)}{2^8}.\tag{32}$$

In (32) if $A_4<\frac{3}{16}$ we get,

$$\frac{\partial\lambda}{\partial l}>0.\tag{33}$$

From (33) we see that when wage rate of the laborers increases, the value of Lagrange multiplier i.e., marginal profit also increases. Hence, as the wage rate increases, laborers work for more working hours to earn more money. Consequently, due to substitution effects the laborers can earn more earnings spending more working hours. As a result, the industry can move to the profit maximization strategies.

In (32) if $A_4>\frac{3}{16}$ we get,

$$\frac{\partial\lambda}{\partial l}<0.\tag{34}$$

From (34) we see that when wage rate of the laborers increases, the value of Lagrange multiplier i.e., marginal profit decreases. It seems that due to income effects as the laborers can earns more money by the less working hours, they remain absent in the industry frequently. As a result, the industry faces unsustainable environment due to shortage of workers.

In (32) if $A_4 = \frac{3}{16}$ we get,

$$\frac{\partial \lambda}{\partial l} = 0 . \tag{35}$$

From (35) we see that when wage rate of the laborers increases, there is no change of value of Lagrange multiplier. It seems that there is no effect on Lagrange multiplier at any change of workers levels for increasing returns to scale.

Now we consider $\alpha = \beta = \gamma = \delta = \frac{1}{4}$ then we get, $\Psi = 1$, i.e., for constant returns to scale, in (31) we get,

$$\frac{\partial \lambda}{\partial l} = \frac{\Gamma^4 \left(73 - 256 A_4\right)}{2^{16} A_2 A_2 A_2 |J|} . \tag{36}$$

In (36) if $A_4 < \frac{73}{256}$ we get,

$$\frac{\partial \lambda}{\partial l} > 0 . \tag{37}$$

From (37) we see that when wage rate of the laborers increases, the value of Lagrange multiplier i.e., marginal profit also increases. Hence, we face the same situation as in (33), and we have observed that increased wage rate becomes boon for the industry.

In (36) if $A_4 > \frac{73}{256}$ we get,

$$\frac{\partial \lambda}{\partial l} < 0 . \tag{38}$$

From (38) we see that when wage rate of the laborers increases, the value of Lagrange multiplier i.e., marginal profit decreases, which is same situation as in (34). In this situation the industry may face unsustainable environment, and through this strategy profit maximization policy may fail.

In (36) if $A_4 = \frac{73}{256}$ we get,

$$\frac{\partial \lambda}{\partial l} = 0 . \tag{39}$$

From (39) we see that when wage rate of the laborers increases, there is no change of the level of Lagrange multiplier. It seems that there is no effect when the wage rate of the laborers increases or decreases.

Now we consider $\alpha = \beta = \gamma = \frac{1}{8}$ and $\delta = \frac{1}{2}$ then we get, $\Psi = \frac{7}{8}$, i.e., for decreasing returns to scale, in (31) we get,

$$\frac{\partial \lambda}{\partial l} = \frac{\Gamma^4}{2^{20} A_1^{\frac{3}{2}} A_2^{\frac{1}{2}} A_3^{\frac{3}{2}} A_4^{\frac{3}{2}} |J|} \left(65 - 1024 A_4\right). \tag{40}$$

In (40) if $A_4 < \frac{65}{1024}$ we get,

$$\frac{\partial \lambda}{\partial l} > 0 . \tag{41}$$

From (41) we see that when wage rate of the laborers increases, the value of Lagrange multiplier i.e., marginal profit also increases. Hence, we face the same situation as in (33). Therefore, the industry is in the highest esteem of profit maximization atmosphere.

In (40) if $A_4 > \frac{65}{1024}$ we get,

$$\frac{\partial \lambda}{\partial l} < 0 . \tag{42}$$

From (42) we see that when wage rate of the laborers increases, the value of Lagrange multiplier i.e., marginal profit decreases, which is the same situation as in (34). In this situation the industry may face unsustainable environment, and through this strategy profit maximization policy may fail.

In (40) if $A_4 = \frac{65}{1024}$ we get,

$$\frac{\partial \lambda}{\partial l} = 0 . \tag{43}$$

From (43) we see that when wage rate of the laborers increases, there is no change of value of Lagrange multiplier. Hence, we face the same situation as in (35), i.e., we observe that there is no relation between wage rate and Lagrange multiplier.

Now we analyze the economic effects on Lagrange multiplier λ when per unit cost of principal raw material increases. Taking T_{13} (i.e., term of 1st row and 3rd column) from both sides of (19) we get (Islam et al., 2011; Roy et al., 2021; Mohajan & Mohajan, 2022c),

$$\frac{\partial \lambda}{\partial m} = \frac{A_3}{J}[C_{11}] + \frac{\lambda}{J}[C_{41}]$$

$$= \frac{A_3}{|J|}\text{Cofactor of } C_{11} + \frac{\lambda}{|J|}\text{Cofactor of C}_{41}$$

$$= \frac{A_3}{|J|}\begin{vmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{vmatrix} - \frac{\lambda}{J}\begin{vmatrix} -B_1 & -B_2 & -B_3 & -B_4 \\ S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{vmatrix}$$

$$= \frac{A_3}{|J|}\left\{ S_{11}\begin{vmatrix} S_{22} & S_{23} & S_{24} \\ S_{32} & S_{33} & S_{34} \\ S_{42} & S_{43} & S_{44} \end{vmatrix} - S_{12}\begin{vmatrix} S_{21} & S_{23} & S_{24} \\ S_{31} & S_{33} & S_{34} \\ S_{41} & S_{43} & S_{44} \end{vmatrix} + S_{13}\begin{vmatrix} S_{21} & S_{22} & S_{24} \\ S_{31} & S_{32} & S_{34} \\ S_{41} & S_{42} & S_{44} \end{vmatrix} - S_{14}\begin{vmatrix} S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \\ S_{41} & S_{42} & S_{43} \end{vmatrix} \right\}$$

$$- \frac{\lambda}{J}\left\{ -B_1\begin{vmatrix} S_{12} & S_{13} & S_{14} \\ S_{22} & S_{23} & S_{24} \\ S_{42} & S_{43} & S_{44} \end{vmatrix} + B_2\begin{vmatrix} S_{11} & S_{13} & S_{14} \\ S_{21} & S_{23} & S_{24} \\ S_{41} & S_{43} & S_{44} \end{vmatrix} - B_3\begin{vmatrix} S_{11} & S_{12} & S_{14} \\ S_{21} & S_{22} & S_{24} \\ S_{41} & S_{42} & S_{44} \end{vmatrix} + B_4\begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{41} & S_{42} & S_{43} \end{vmatrix} \right\}$$

$$= \frac{A_3}{|J|}\left[-S_{11}\left\{ S_{22}\left(S_{33}S_{44} - S_{43}S_{34}\right) + S_{23}\left(S_{42}S_{34} - S_{32}S_{44}\right) + S_{24}\left(S_{32}S_{43} - S_{42}S_{33}\right)\right\}\right.$$

$$- S_{12}\left\{ -S_{21}\left(S_{33}S_{44} - S_{43}S_{34}\right) + S_{23}\left(-S_{41}S_{34} + S_{31}S_{44}\right) + S_{24}\left(-S_{31}S_{43} + S_{41}S_{33}\right)\right\}$$

$$+S_{13}\{-S_{21}(S_{32}S_{44}-S_{42}S_{34})+S_{22}(-S_{41}S_{34}+S_{31}S_{44})+S_{24}(-S_{31}S_{42}+S_{41}S_{32})\}$$

$$-S_{14}\{-S_{21}(S_{32}S_{43}-S_{42}S_{33})+S_{22}(-S_{41}S_{33}+S_{31}S_{43})+S_{23}(-S_{31}S_{42}+S_{41}S_{32})\}]$$

$$-\frac{\lambda}{|J|}[-B_1\{S_{12}(S_{23}S_{44}-S_{43}S_{24})+S_{13}(S_{42}S_{24}-S_{22}S_{44})+S_{14}(S_{22}S_{43}-S_{42}S_{23})\}$$

$$+B_2\{S_{11}(S_{23}S_{44}-S_{43}S_{24})+S_{13}(S_{41}S_{24}-S_{21}S_{44})+S_{14}(-S_{21}S_{43}+S_{41}S_{23})\}$$

$$-B_3\{S_{11}(S_{22}S_{44}-S_{42}S_{24})+S_{12}(S_{41}S_{24}-S_{21}S_{44})+S_{14}(S_{21}S_{42}-S_{41}S_{22})\}$$

$$+B_4\{S_{11}(S_{22}S_{43}-S_{42}S_{23})+S_{12}(S_{41}S_{23}-S_{21}S_{43})+S_{13}(S_{21}S_{42}-S_{41}S_{22})\}]$$

$$=\frac{A_3}{|J|}\{-S_{11}S_{22}S_{33}S_{44}+S_{11}S_{24}S_{42}S_{33}-S_{11}S_{24}S_{32}S_{43}+S_{11}S_{23}S_{32}S_{44}-S_{11}S_{23}S_{42}S_{34}+S_{11}S_{22}S_{43}S_{34}$$

$$+S_{12}S_{21}S_{33}S_{44}-S_{12}S_{21}S_{43}S_{34}+S_{12}S_{23}S_{41}S_{34}-S_{12}S_{23}S_{31}S_{44}+S_{12}S_{24}S_{31}S_{43}-S_{12}S_{24}S_{41}S_{33}$$

$$-S_{13}S_{21}S_{32}S_{44}+S_{13}S_{21}S_{42}S_{34}-S_{13}S_{22}S_{41}S_{34}+S_{13}S_{22}S_{31}S_{44}-S_{13}S_{24}S_{31}S_{42}+S_{13}S_{24}S_{41}S_{32}$$

$$+S_{14}S_{21}S_{32}S_{43}-S_{14}S_{21}S_{42}S_{33}+S_{14}S_{22}S_{41}S_{33}-S_{14}S_{22}S_{31}S_{43}+S_{14}S_{23}S_{31}S_{42}-S_{14}S_{23}S_{41}S_{32}\}$$

$$-\frac{\lambda}{|J|}\{-B_1S_{12}S_{23}S_{44}+B_1S_{12}S_{43}S_{24}-B_1S_{13}S_{42}S_{24}+B_1S_{13}S_{22}S_{44}-B_1S_{14}S_{22}S_{43}+B_1S_{14}S_{42}S_{23}$$

$$+B_2S_{11}S_{23}S_{44}-B_2S_{11}S_{43}S_{24}+B_2S_{13}S_{41}S_{24}-B_2S_{13}S_{21}S_{44}+B_2S_{14}S_{21}S_{43}-B_2S_{14}S_{41}S_{23}$$

$$-B_3S_{11}S_{22}S_{44}+B_3S_{11}S_{42}S_{24}-B_3S_{12}S_{41}S_{24}+B_3S_{12}S_{21}S_{44}-B_3S_{14}S_{21}S_{42}+B_3S_{14}S_{41}S_{22}$$

$$+B_4S_{11}S_{22}S_{43}-B_4S_{11}S_{42}S_{23}+B_4S_{12}S_{41}S_{23}-B_4S_{12}S_{21}S_{43}+B_4S_{13}S_{21}S_{42}-B_4S_{13}S_{41}S_{22}\}$$

$$=\frac{1}{|J|}\frac{\Gamma^4A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2^2A_3A_4^2}\{-\alpha(\alpha-1)\beta(\beta-1)\gamma(\gamma-1)\delta(\delta-1)+\alpha(\alpha-1)\beta^2\gamma(\gamma-1)\delta^2$$

$$-\alpha(\alpha-1)\beta^2\gamma^2\delta^2+\alpha(\alpha-1)\beta^2\gamma^2\delta(\delta-1)-\alpha(\alpha-1)\beta^2\gamma^2\delta^2+\alpha(\alpha-1)\beta(\beta-1)\gamma^2\delta^2$$

$$+\alpha^2\beta^2\gamma(\gamma-1)\delta(\delta-1)-\alpha^2\beta^2\gamma^2\delta^2+\alpha^2\beta^2\gamma^2\delta^2-\alpha^2\beta^2\gamma^2\delta(\delta-1)+\alpha^2\beta^2\gamma^2\delta^2$$

$$-\alpha^2\beta^2\gamma(\gamma-1)\delta^2-\alpha^2\beta^2\gamma^2\delta(\delta-1)+\alpha^2\beta^2\gamma^2\delta^2-\alpha^2\beta(\beta-1)\gamma^2\delta^2+\alpha^2\beta(\beta-1)\gamma^2\delta(\delta-1)$$

$$-\alpha^2\beta^2\gamma^2\delta^2 \quad +\alpha^2\beta^2\gamma^2\delta^2 \quad +\alpha^2\beta^2\gamma^2\delta^2 \quad -\alpha^2\beta^2\gamma(\gamma-1)\delta^2 \quad -\alpha^2\beta(\beta-1)\gamma^2\delta^2 \quad +\alpha^2\beta^2\gamma^2\delta^2$$

$$-\alpha^2\beta^2\gamma^2\delta^2\Big\} \; -\frac{\lambda}{|J|}\frac{\Gamma^4 A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2^2A_3^2A_4^2}$$

$$\Big\{-kA_1A_3\alpha\beta^2\gamma\delta(\delta-1) \; +kA_1A_3\alpha\beta^2\gamma\delta^2 \; -kA_1A_3\alpha\beta^2\gamma\delta^2 \; +kA_1A_3\alpha\beta(\beta-1)\gamma\delta(\delta-1) \; -kA_1A_3\alpha\beta^2\gamma\delta^2$$

$$+kA_1A_3\alpha\beta^2\gamma\delta^2 \quad +lA_2A_3\alpha(\alpha-1)\beta\gamma\delta(\delta-1) \quad -lA_2A_3\alpha(\alpha-1)\beta\gamma\delta^2 \quad +lA_2A_3\alpha^2\beta\gamma\delta^2$$

$$-lA_2A_3\alpha^2\beta\gamma\delta(\delta-1) \quad +lA_2A_3\alpha^2\beta\gamma\delta^2 \quad -lA_2A_3\alpha^2\beta\gamma\delta^2 \quad -mA_3^2\alpha(\alpha-1)\beta(\beta-1)\delta(\delta-1)$$

$$+mA_3^2\alpha(\alpha-1)\beta^2\delta^2 \quad -mA_3^2\alpha^2\beta^2\delta^2 \quad +mA_3^2\alpha^2\beta^2\delta(\delta-1) \quad -mA_3^2\alpha^2\beta^2\delta^2 \quad +mA_3^2\alpha^2\beta(\beta-1)\delta^2$$

$$+nA_3A_4\alpha(\alpha-1)\beta(\beta-1)\gamma\delta \quad -nA_3A_4\alpha(\alpha-1)\beta^2\gamma\delta \quad +nA_3A_4\alpha^2\beta^2\gamma\delta \quad -nA_3A_4\alpha^2\beta^2\gamma\delta$$

$$+nA_3A_4\alpha^2\beta^2\gamma\delta - nA_3A_4\alpha^2\beta(\beta-1)\gamma\delta\Big\}$$

$$=\frac{1}{|J|}\frac{\Gamma^4\alpha\beta\gamma\delta A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2^2A_3A_4^2} \quad \Big\{-(\alpha-1)(\beta-1)(\gamma-1)(\delta-1) \; +(\alpha-1)\beta\gamma(\delta-1) \; +\alpha\beta(\gamma-1)(\delta-1)$$

$$-2\alpha\beta\gamma(\delta-1) \; +\alpha(\beta-1)\gamma(\delta-1) \; -2\alpha\beta(\gamma-1)\delta \; +3\alpha\beta\gamma\delta \; -2(\alpha-1)\beta\gamma\delta \; +(\alpha-1)(\beta-1)\gamma\delta$$

$$+(\alpha-1)\beta(\gamma-1)\delta \quad -3\alpha(\beta-1)\gamma\delta\Big\} \quad -\frac{1}{|J|}\frac{\Gamma^3\alpha\beta\gamma\delta A_1^{3\alpha}A_2^{3\beta}A_3^{3\gamma}A_4^{3\delta}}{A_1^2A_2^2A_3A_4^2}\frac{\Gamma A_1^{\alpha}A_1^{\beta}A_1^{\gamma}A_1^{\delta}\Psi}{B}\Big\{-\alpha\beta(\delta-1)$$

$$+\alpha(\beta-1)(\delta-1) \quad +(\alpha-1)\beta(\delta-1) \quad -(\alpha-1)(\beta-1)(\delta-1) \quad -\alpha\beta\delta \quad +\alpha(\beta-1)\delta$$

$$+(2A_4-1)(\alpha-1)(\beta-1)\delta-(2A_4-1)(\alpha-1)\beta\delta+(2A_4-1)\alpha\beta\delta-(2A_4-1)\alpha(\beta-1)\delta\Big\}$$

$$\frac{\partial\lambda}{\partial m}=\frac{1}{|J|}\frac{\Gamma^4\alpha\beta\gamma\delta A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2^2A_3A_4^2}\left(2A_4\delta-2\alpha\beta\gamma\delta+2\alpha\gamma\delta+\alpha\beta\delta+\delta+\alpha+\beta+\gamma-1\right). \tag{44}$$

Now we consider $\alpha=\beta=\gamma=\delta=\frac{1}{2}$ then we get, $\Psi=2$, i.e., for increasing returns to scale, in (44) we get,

$$\frac{\partial\lambda}{\partial m}=\frac{1}{|J|}\frac{\Gamma^4 A_3}{2^6}\left(4A_4+5\right)>0. \tag{45}$$

From the relation (45) we see that when per unit cost of principal raw material increases, the level of Lagrange multiplier, i.e., marginal profit also increases. It seems that the industry is in profit maximization, and its products are also increasing despite increase of cost of principal raw material.

Now we consider $\alpha=\beta=\gamma=\delta=\frac{1}{4}$ then we get, $\Psi=1$, i.e., for constant returns to scale, in (44) we get,

$$\frac{\partial \lambda}{\partial m} = \frac{1}{|J|} \frac{\Gamma^4}{2^6 A_1 A_2 A_4} (64 A_4 + 5) > 0 . \tag{46}$$

From the relation (46) we see that when per unit cost of principal raw material increases, the level of Lagrange multiplier, i.e., marginal profit also increases. It seems that this case is same as (45), i.e., both increasing and constant returns to scale may present the industry sustainable environment.

Now we consider $\alpha = \beta = \gamma = \frac{1}{8}$ and $\delta = \frac{1}{2}$, then we get, $\Psi = \frac{7}{8} < 1$, i.e., for decreasing returns to scale, in (44) we get,

$$\frac{\partial \lambda}{\partial m} = \frac{1}{|J|} \frac{\Gamma^4}{2^{19} A_1^{\frac{3}{2}} A_2^{\frac{3}{2}} A_3^{\frac{1}{2}} A_4^{\frac{3}{2}}} (512 A_4 + 153) > 0 . \tag{47}$$

From the relation (47) we see have obtained the same result as there in (45) and (46). Hence, the industry is in sustainable profit maximization stage at any situation.

Now we analyze the economic effects on Lagrange multiplier when the discounted price of the irregular raw material, n_0 increases. Taking T_{14} (i.e., term of 1st row and 4th column) from both sides of (19) we get (Moolio et al., 2009; Wiese, 2021; Mohajan & Mohajan, 2023c),

$$\frac{\partial \lambda}{\partial n_0} = \frac{A_4^2}{|J|}[C_{11}] - \frac{A_4}{|J|}[C_{11}] + 2\lambda \frac{A_4}{|J|}[C_{51}] - \frac{\lambda}{|J|}[C_{51}]$$

$$= \frac{A_4^2}{|J|}\text{Cofactor of } C_{11} - \frac{A_4}{|J|}\text{Cofactor of } C_{11} + 2\lambda \frac{A_4}{|J|}\text{Cofactor of } \mathrm{C}_{51} - \frac{\lambda}{|J|}\text{Cofactor of } \mathrm{C}_{51}$$

$$= \frac{A_4^2 - A_4}{|J|} \begin{vmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{vmatrix} - \frac{\lambda(1 - 2A_4)}{|J|} \begin{vmatrix} -B_1 & -B_2 & -B_3 & -B_4 \\ S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \end{vmatrix}$$

$$= \frac{A_4^2 - A_4}{|J|} \left\{ S_{11} \begin{vmatrix} S_{22} & S_{23} & S_{24} \\ S_{32} & S_{33} & S_{34} \\ S_{42} & S_{43} & S_{44} \end{vmatrix} - S_{12} \begin{vmatrix} S_{21} & S_{23} & S_{24} \\ S_{31} & S_{33} & S_{34} \\ S_{41} & S_{43} & S_{44} \end{vmatrix} + S_{13} \begin{vmatrix} S_{21} & S_{22} & S_{24} \\ S_{31} & S_{32} & S_{34} \\ S_{41} & S_{42} & S_{44} \end{vmatrix} - S_{14} \begin{vmatrix} S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \\ S_{41} & S_{42} & S_{43} \end{vmatrix} \right\}$$

$$- \frac{\lambda(1 - 2A_4)}{|J|} \left\{ -B_1 \begin{vmatrix} S_{12} & S_{13} & S_{14} \\ S_{22} & S_{23} & S_{24} \\ S_{32} & S_{33} & S_{34} \end{vmatrix} + B_2 \begin{vmatrix} S_{11} & S_{13} & S_{14} \\ S_{21} & S_{23} & S_{24} \\ S_{31} & S_{33} & S_{34} \end{vmatrix} - B_3 \begin{vmatrix} S_{11} & S_{12} & S_{14} \\ S_{21} & S_{22} & S_{24} \\ S_{31} & S_{32} & S_{34} \end{vmatrix} + B_4 \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{vmatrix} \right\}$$

$$= \frac{A_4^2 - A_4}{|J|} [\ S_{11} \{ S_{22}(S_{33}S_{44} - S_{43}S_{34}) + S_{23}(S_{42}S_{34} - S_{32}S_{44}) + S_{24}(S_{32}S_{43} - S_{42}S_{33}) \}$$

$$- S_{12} \{ -S_{21}(S_{33}S_{44} - S_{43}S_{34}) + S_{23}(-S_{41}S_{34} + S_{31}S_{44}) + S_{24}(-S_{31}S_{43} + S_{41}S_{33}) \}$$

$$+S_{13}\{-S_{21}(S_{32}S_{44}-S_{42}S_{34})+S_{22}(-S_{41}S_{34}+S_{31}S_{44})+S_{24}(-S_{31}S_{42}+S_{41}S_{32})\}$$

$$-S_{14}\{-S_{21}(S_{32}S_{43}-S_{42}S_{33})+S_{22}(-S_{41}S_{33}+S_{31}S_{43})+S_{23}(-S_{31}S_{42}+S_{41}S_{32})\}]$$

$$-\frac{\lambda(1-2A_4)}{|J|}[-B_1\{-S_{12}(S_{23}S_{34}-S_{33}S_{24})+S_{13}(-S_{32}S_{24}+S_{22}S_{34})+S_{14}(-S_{22}S_{33}+S_{32}S_{23})\}$$

$$+B_2\{-S_{11}(S_{23}S_{34}-S_{33}S_{24})+S_{13}(-S_{31}S_{24}+S_{21}S_{34})+S_{14}(-S_{21}S_{33}+S_{31}S_{23})\}$$

$$-B_3\{-S_{11}(S_{22}S_{34}-S_{32}S_{24})+S_{12}(-S_{31}S_{24}+S_{21}S_{34})+S_{14}(-S_{21}S_{32}+S_{31}S_{22})\}$$

$$+B_4\{-S_{11}(S_{22}S_{33}-S_{32}S_{23})+S_{12}(-S_{31}S_{23}+S_{21}S_{33})+S_{13}(-S_{21}S_{32}+S_{31}S_{22})\}]$$

$$=\frac{A_4^2-A_4}{J}\{-S_{11}S_{22}S_{33}S_{44}\quad +S_{11}S_{24}S_{42}S_{33}\quad -S_{11}S_{24}S_{32}S_{43}\quad +S_{11}S_{23}S_{32}S_{44}\quad -S_{11}S_{23}S_{42}S_{34}$$

$$+S_{11}S_{22}S_{43}S_{34}\quad +S_{12}S_{21}S_{33}S_{44}\quad -S_{12}S_{21}S_{43}S_{34}\quad +S_{12}S_{23}S_{41}S_{34}\quad -S_{12}S_{23}S_{31}S_{44}\quad +S_{12}S_{24}S_{31}S_{43}$$

$$-S_{12}S_{24}S_{41}S_{33}\quad -S_{13}S_{21}S_{32}S_{44}\quad +S_{13}S_{21}S_{42}S_{34}\quad -S_{13}S_{22}S_{41}S_{34}\quad +S_{13}S_{22}S_{31}S_{44}\quad -S_{13}S_{24}S_{31}S_{42}$$

$$+S_{13}S_{24}S_{41}S_{32}\quad +S_{14}S_{21}S_{32}S_{43}\quad -S_{14}S_{21}S_{42}S_{33}\quad +S_{14}S_{22}S_{41}S_{33}\quad -S_{14}S_{22}S_{31}S_{43}\quad +S_{14}S_{23}S_{31}S_{42}$$

$$-S_{14}S_{23}S_{41}S_{32}\}\ +\frac{\lambda(1-2A_4)}{|J|}\{B_1S_{12}S_{23}S_{34}\ -B_1S_{12}S_{33}S_{24}\ +B_1S_{13}S_{32}S_{24}\ -B_1S_{13}S_{22}S_{34}\ +B_1S_{14}S_{22}S_{33}$$

$$-B_1S_{14}S_{32}S_{23}\ -B_2S_{11}S_{23}S_{34}\ +B_2S_{11}S_{33}S_{24}\ -B_2S_{13}S_{31}S_{24}\ +B_2S_{13}S_{21}S_{34}\ -B_2S_{14}S_{21}S_{33}\ +B_2S_{14}S_{31}S_{23}$$

$$+B_3S_{11}S_{22}S_{34}\ -B_3S_{11}S_{32}S_{24}\ +B_3S_{12}S_{31}S_{24}\ -B_3S_{12}S_{21}S_{34}\ +B_3S_{14}S_{21}S_{32}\ -B_3S_{14}S_{31}S_{22}\ -B_4S_{11}S_{22}S_{33}$$

$$+B_4S_{11}S_{32}S_{23}\ -B_4S_{12}S_{31}S_{23}\ +B_4S_{12}S_{21}S_{33}\ -B_4S_{13}S_{21}S_{32}\ +B_4S_{13}S_{31}S_{22}\}$$

$$=-\frac{A_4^2-A_4}{|J|}\frac{\Gamma^4A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2^2A_3^2A_4^2}\{-\alpha(\alpha-1)\beta(\beta-1)\gamma(\gamma-1)\delta(\delta-1)\qquad +\alpha(\alpha-1)\beta^2\gamma(\gamma-1)\delta^2$$

$$-\alpha(\alpha-1)\beta^2\gamma^2\delta^2\quad +\alpha(\alpha-1)\beta^2\gamma^2\delta(\delta-1)\quad -\alpha(\alpha-1)\beta^2\gamma^2\delta^2\quad +\alpha(\alpha-1)\beta(\beta-1)\gamma^2\delta^2$$

$$+\alpha^2\beta^2\gamma(\gamma-1)\delta(\delta-1)\quad -\alpha^2\beta^2\gamma^2\delta^2\quad +\alpha^2\beta^2\gamma^2\delta^2\quad -\alpha^2\beta^2\gamma^2\delta(\delta-1)\quad +\alpha^2\beta^2\gamma^2\delta^2$$

$$-\alpha^2\beta^2\gamma(\gamma-1)\delta^2\ -\alpha^2\beta^2\gamma^2\delta(\delta-1)\ +\alpha^2\beta^2\gamma^2\delta^2\ -\alpha^2\beta(\beta-1)\gamma^2\delta^2\ +\alpha^2\beta(\beta-1)\gamma^2\delta(\delta-1)$$

$$-\alpha^2\beta^2\gamma^2\delta^2\ +\alpha^2\beta^2\gamma^2\delta^2\ +\alpha^2\beta^2\gamma^2\delta^2\ -\alpha^2\beta^2\gamma(\gamma-1)\delta^2\ -\alpha^2\beta(\beta-1)\gamma^2\delta^2\ +\alpha^2\beta^2\gamma^2\delta^2$$

$$-\alpha^2\beta^2\gamma^2\delta^2\Big\} + \frac{(1-2A_4)}{|J|}\frac{\Gamma^{34}A_1^{3\alpha}A_2^{3\beta}A_3^{3\gamma}A_4^{3\delta}}{A_1^2A_2^2A_3^2A_4^2}\frac{\Gamma A_1^{\alpha}A_1^{\beta}A_1^{\gamma}A_1^{\delta}\Psi}{B}\Big\{kA_1A_4\alpha\beta^2\gamma^2\delta - kA_1A_4\alpha\beta^2\gamma(\gamma-1)\delta$$

$$+kA_1A_4\alpha\beta^2\gamma^2\delta - kA_1A_4\alpha\beta(\beta-1)\gamma^2\delta + kA_1A_4\alpha\beta(\beta-1)\gamma(\gamma-1)\delta - kA_1A_4\alpha\beta^2\gamma^2\delta$$

$$-lA_2A_4\alpha(\alpha-1)\beta^2\gamma^2\delta + lA_2A_4\alpha(\alpha-1)\beta\gamma(\gamma-1)\delta - lA_2A_4\alpha^2\beta\gamma^2\delta + lA_2A_4\alpha^2\beta\gamma^2\delta$$

$$-lA_2A_4\alpha^2\beta\gamma(\gamma-1)\delta + lA_2A_4\alpha^2\beta\gamma^2\delta + mA_3A_4\alpha(\alpha-1)\beta(\beta-1)\gamma\delta - mA_3A_4\alpha(\alpha-1)\beta^2\gamma\delta$$

$$+mA_3A_4\alpha^2\beta^2\gamma\delta - mA_3A_4\alpha^2\beta^2\gamma\delta + mA_3A_4\alpha^2\beta^2\gamma\delta - mA_3A_4\alpha^2\beta(\beta-1)\gamma\delta$$

$$-nA_4^2\alpha(\alpha-1)\beta(\beta-1)\gamma(\gamma-1) + nA_4^2\alpha(\alpha-1)\beta^2\gamma^2 - nA_4^2\alpha^2\beta^2\gamma^2 + nA_4^2\alpha^2\beta^2\gamma(\gamma-1) - nA_4^2\alpha^2\beta^2\gamma^2$$

$$+nA_4^2\alpha^2\beta(\beta-1)\gamma^2\Big\}$$

$$= -\frac{A_4^2 - A_4}{|J|}\frac{\alpha\beta\gamma\delta\,\Gamma^4A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2^2A_3^2A_4^2}\Big\{-(\alpha-1)(\beta-1)(\gamma-1)(\delta-1) + (\alpha-1)\beta\gamma(\delta-1)$$

$$+\alpha\beta(\gamma-1)(\delta-1) - 2\alpha\beta\gamma(\delta-1) + \alpha(\beta-1)\gamma(\delta-1) - 2\alpha\beta(\gamma-1)\delta + 3\alpha\beta\gamma\delta - 2(\alpha-1)\beta\gamma\delta$$

$$+(\alpha-1)(\beta-1)\gamma\delta + (\alpha-1)\beta(\gamma-1)\delta - 3\alpha(\beta-1)\gamma\delta + \frac{\alpha\beta\gamma\delta(A_4-2A_4^2)}{|J|}\frac{\Gamma^4A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2^2A_3^2A_4^2}\Big\{3\alpha\beta\gamma$$

$$-2(\alpha-1)\beta\gamma + (\alpha-1)(\beta-1)\gamma - 2\alpha(\beta-1)\gamma - 2\alpha\beta(\gamma-1) + \alpha(\beta-1)(\gamma-1) + (\alpha-1)\beta(\gamma-1)$$

$$-(2A_4-1)(\alpha-1)(\beta-1)(\gamma-1) + (2A_4-1)(\alpha-1)\beta\gamma + (2A_4-1)\alpha\beta(\gamma-1) - 2(2A_4-1)\alpha\beta\gamma$$

$$+(2A_4-1)\alpha(\beta-1)\gamma\Big\}$$

$$\frac{\partial\lambda}{\partial n_0} = -\frac{A_4-1}{|J|}\frac{\alpha\beta\gamma\delta\,\Gamma^4A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2^2A_3^2A_4}\{2\alpha\beta\delta + 2\alpha\gamma\delta - \alpha\delta - \alpha\beta - \alpha\gamma + \alpha + \beta + \gamma + \delta - 1\}$$

$$+\frac{\alpha\beta\gamma\delta(1-2A_4)}{|J|}\frac{\Gamma^4A_1^{4\alpha}A_2^{4\beta}A_3^{4\gamma}A_4^{4\delta}}{A_1^2A_2^2A_3^2A_4}\{(2\alpha+2\beta+2\gamma-1) - A_4(2\alpha+2\beta+2\gamma-2)\}. \quad (48)$$

Now we consider $\alpha=\beta=\gamma=\delta=\frac{1}{4}$ then we get, $\Psi = 2$, i.e., for increasing returns to scale, in (48) we get,

$$\frac{\partial\lambda}{\partial n_0} = \frac{\Gamma^4}{2^{11}A_1A_2A_3}\{(A_4-1)(8A_4-3)\}. \quad (49)$$

In equation (49) if $A_4 > 1$ or $A_4 < \frac{3}{8}$ we get,

$$\frac{\partial \lambda}{\partial n_0} > 0 . \tag{50}$$

From the inequality (50) we see that when discounted price of the irregular raw material increases, the level of Lagrange multiplier also increases. It seems that although cost of irregular raw material increases, the demand of the products of the industry is also increased in the society, and the industry increases its production level. We believe that for increasing returns to scale the industry is in sustainable environment and profit maximization is possible for this industry.

In equation (49) if $A_4 < 1$ and $A_4 > \frac{3}{8}$ we get,

$$\frac{\partial \lambda}{\partial n_0} < 0 . \tag{51}$$

From the inequality (51) we see that when discounted price of the irregular raw material, the Lagrange multiplier, i.e., the marginal profit is decreased. It seems that irregular raw material is essential for the firm and more capital is used for purchasing the irregular raw material. Consequently, in this situation profit maximization atmosphere will be difficult for this industry.

In equation (49) if $A_4 = 1$ or $A_4 = \frac{3}{8}$ we get,

$$\frac{\partial \lambda}{\partial n_0} = 0 . \tag{52}$$

From the equation (52) we see that when discounted price of the irregular raw material increases, there is no change of Lagrange multiplier. In this situation, it seems that increase or decrease price of the irregular raw material will not affect the level of Lagrange multiplier.

Now we consider $\alpha = \beta = \gamma = \delta = \frac{1}{2}$ then we get, $\Psi = 1$, i.e., for constant returns to scale, in (48) we get,

$$\frac{\partial \lambda}{\partial n_0} = \frac{\Gamma^4 3\left(A_4^2 - A_4\right)}{2^6 |J|} + \frac{\Gamma^4 \left(A_4 - 2A_4^2\right)}{2^4 |J|}\left(2 - A_4\right)$$

$$\frac{\partial \lambda}{\partial n_0} = \frac{\Gamma^4}{2^6 |J|}\left(A_4^2 - \frac{1}{2}\right). \tag{53}$$

In equation (53) if $A_4 > \pm \frac{1}{\sqrt{2}}$ we get,

$$\frac{\partial \lambda}{\partial n_0} > 0 . \tag{54}$$

The inequality (54) provides the same result as in (50) for constant returns to scale. Hence, we see that the industry is in sustainable condition for constant returns to scale.

In equation (53) if $0 < A_4 < \frac{1}{\sqrt{2}}$ we get,

$$\frac{\partial \lambda}{\partial n_0} < 0. \tag{55}$$

The inequality (55) provides the same result as in (51) for constant returns to scale. Hence, it is observed that the industry is in difficulties to reach in sustainable condition.

In equation (53) if $A_4 = \frac{1}{\sqrt{2}}$ we get,

$$\frac{\partial \lambda}{\partial n_0} = 0. \tag{56}$$

The equation (56) provides the same result as in (52) for constant returns to scale. That is, in brief, the industry is indifferent about Lagrange multiplier about the case of irregular raw material.

Now we consider $\alpha = \beta = \gamma = \frac{1}{8}$ and $\delta = \frac{1}{2}$, then we get, $\Psi = \frac{7}{8} < 1$, i.e., for decreasing returns to scale, in (48) we get,

$$\frac{\partial \lambda}{\partial n_0} = \frac{\Gamma^4}{2^{14} A_1^{\frac{3}{2}} A_2^{\frac{3}{2}} A_3^{\frac{3}{2}} A_4^{\frac{1}{2}} |J|} \{3(A_4 - 1) + 4(5A_4 + 1)(2A_4 - 1)\}$$

$$\frac{\partial \lambda}{\partial n_0} = \frac{\Gamma^4}{2^{15} \times 5 A_1^{\frac{3}{2}} A_2^{\frac{3}{2}} A_3^{\frac{3}{2}} A_4^{\frac{1}{2}} |J|} \left\{ \left(2A_4 - \frac{9}{40}\right)^2 - \frac{280}{81} \right\}. \tag{57}$$

In equation (57) if $A_4 > \left(81 \pm \sqrt{70}\right)/720$ we get,

$$\frac{\partial \lambda}{\partial n_0} > 0. \tag{58}$$

The inequality (58) provides the same result as in (50) for decreasing returns to scale. Hence, in this situation it seems that the industry is in better position in the economic sustainability.

In equation (57) if $A_4 < \left(81 \pm \sqrt{70}\right)/720$ we get,

$$\frac{\partial \lambda}{\partial n_0} < 0. \tag{59}$$

The inequality (59) provides the same result as in (51) for decreasing returns to scale. It seems that the industry may face various complications during profit maximization attempts.

In equation (57) if $A_4 = \left(81 \pm \sqrt{70}\right)/720$ we get,

$$\frac{\partial \lambda}{\partial n_0} = 0. \tag{60}$$

The equation (60) provides the same result as in (52) for decreasing returns to scale. In this stage we see that the industry can increase or decrease purchasing irregular raw material, as it does not affect the level of Lagrange multiplier.

11. Conclusions

In this study we have discussed the economic effects of Lagrange multiplier if costs of various inputs of an industry increase. We have considered here nonlinear budget constraint to provide economic predictions when we have searched a sustainable environment for an industry. The article is started with Cobb-Douglas productions function as profit function. We have also used 5×5 bordered Hessian matrix and 5×5 Jacobian to operate the mathematical formulations efficiently.

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