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Some Integral Representations, Transformations, and Differential Relations for Mittag-Leffler Function of Double Variables

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Abstract

The fundamental purpose of this research is to offer a natural progression in the mathematical properties and presentation of specific Mittag-Leffler-type functions with two variables. Integral representations, differential relations, and integral transforms are established. Special cases and the consequences of our main findings are also highlighted.

Keywords: Mittag-Leffler type functions, fractional calculus operators, integral transforms, integral relations, operational relations

1. Introduction and Definitions

The Mittag-Leffler function has gained importance and popularity due to its applications in the solution of fractional order differential equations and fractional order integral equations (see for example (Gorenflo R, Kilbas A.A. & Rogosin S.V., 1998; Gorenflo R & Mainardi R., 2000; Gorenflo R., Kilbas A.A., Mainardi F. & Rogosin S., 2020; Kilbas A.A., Saigo M. & Saxena R.K., 2004; Kilbas A.A. & Saigo M., 1996; Li Z., Liu Y. & Yamamoto M., 2015; Luchko Y., 2011; Mittag-Leffler G.M., 1903; Prabhakar T.R., 1971; Saigo M & Kilbas A.A., 1998; Salim T.O., 2009; Salim T.O. & Faraj A.W., 2012; Samko S.G., Kilbas A.A. & Marichev O.I., 1993; Shukla A.K. & Prajapati J.C., 2007; Srivastava H.M. & Manocha H.L., 1984; Wiman A, 1905)). Also, the Mittag-Leffler function plays an important role in various branches of applied mathematics and engineering sciences, such as chemistry, biology, statistics, thermodynamics, mechanics, quantum physics, informatics, and signal processing.

Besides this, the Mittag-Leffler function of several variables appears in the solution of certain boundary value problems involving fractional equations of Volterra type (Samko S.G., Kilbas A.A. & Marichev O.I., 1993), initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation (Luchko Y., 2011), and initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients (Li Z., Liu Y. & Yamamoto M., 2015). In the usual notation $\Gamma(x)$ for the Gamma function and $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$, $n \geq 0, \gamma \neq 0, -1, -2, \cdots$, the Pochhammer symbol. Mittag-Leffler introduced the function $E_\alpha(z)$ in the form (Mittag-Leffler G.M., 1903):

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, (\alpha > 0, \quad z \in \mathbb{C}). \tag{1.1}$$

Note that, the Mittag-Leffler function is a direct generalization of the exponential e^x function to which it reduces for α=1. In (1905) Wiman introduced a generalization of $E_\alpha(z)$ with two parameters in the form:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{1.2}$$

$$(\alpha,\beta\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0).$$

Prabhakar (1971) introduced the function $E_{\alpha,\beta}^{\gamma}(z)$ of their parameter.

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n\, z^n}{\Gamma(\alpha n+\beta)n!}, \tag{1.3}$$

$$(\alpha,\beta,\gamma\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0).$$

Further, three interesting unifications and generalizations of the function $\mathrm{E}_{\alpha}(z)$ were considered by Shukla and Prajapti (2007).

$$E_{\alpha,\beta}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{nq}\, z^{\mathrm{n}}}{\Gamma(\alpha n+\beta)n!}, \tag{1.4}$$

$$(\alpha,\beta,\gamma\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,q\in(0,1)),$$

Salim (2009)

$$E_{\alpha,\beta}^{\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n\, z^{\mathrm{n}}}{\Gamma(\alpha n+\beta)(\delta)_n}, \tag{1.5}$$

$$(\alpha,\beta,\gamma\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\delta>0),$$

and Salim and Faraj (2012)

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{nq}\, z^{\mathrm{n}}}{\Gamma(\alpha n+\beta)(\delta)_{np}}, \tag{1.6}$$

$$(\alpha,\beta,\gamma\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0.\Re(\gamma)>0,\delta>0,(p.q)>0,q\leq\Re(\alpha)+p).$$

For our present study, we recall the following Mellin-Barnes integral representation for the function $E_{\alpha,\beta}^{\gamma}(z)$ (Prabhakar T.R., 1971)

$$E_{\alpha,\beta}^{\gamma}(z)=\frac{1}{2\pi i}\frac{1}{\Gamma(\gamma)}\int_{L}\frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}ds, \tag{1.7}$$

$$(|\arg z|<\pi,\alpha\in\mathbb{R}^{+},\gamma\in\mathbb{C}\setminus\mathbb{Z}_0^{-},\Re(\beta)>0).$$

Mehamet and Banu (2014) introduced an extended Mittag-Leffler function as follows:

$$E_{\alpha,\beta}^{(\gamma;c)}(z;p)=\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\gamma+n,c-\gamma)}{\mathrm{B}(\gamma,c-\gamma)}\frac{(c)_n}{\Gamma(\alpha n+\beta)}\frac{z^{\mathrm{n}}}{n!}, \tag{1.8}$$

$$(p\geq 0;\Re(c)>\Re(\gamma)>0).$$

where $\mathrm{B}_p(x,y)$ is the extended Euler's Beta function defined in (Chaudhry M.A., Qadir A, Srivastava H.M. & Paris R.B., 2004) (see also (Chaudhry M.A. & Zubair S. M., 1994)) as:

$$\mathrm{B}_p(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}dt$$

$$(\Re(p)>0,\Re(x)>0,\Re(y)>0).$$

Gauhar Rahman et al. (2017) introduced the following further generalization of the Mittag-Leffler function.

$$E_{\alpha,\beta}^{\gamma;q,c}(z)=\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\gamma+nq,c-\gamma)\,(c)_{nq}}{\mathrm{B}(\gamma,C-\gamma)\,\Gamma(\alpha n+\beta)}\frac{z^{\mathrm{n}}}{n!},\quad (z,\beta,\gamma\in\mathbb{C}). \tag{1.9}$$

Very recently Bin-Saad et al. (2021) introduced and studied the following Mittag-Leffler function of two variables:

$$E_{\alpha,\beta,\gamma}(z_1,z_2)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{z_1^m z_2^{n-m}}{\Gamma(\alpha m+(n-m)\gamma+\beta)}, \tag{1.10}$$

$$(\Re(\beta)>\Re(\gamma)>\Re(\alpha)>0).$$

Mehmet and Cemaliye (2019):

$$E^{\gamma}_{\alpha,\beta,k}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}}{\Gamma(\alpha+r)\,\Gamma(\beta+ks)} \frac{x^r}{r!}\frac{y^{ks}}{s!}, \tag{1.11}$$

$$(\Re(\alpha),\Re(\beta),\Re(k) > 0).$$

It may be of interest to point out that the series representation (1.11), in particular yields the following relationship.

(i) For $s \mapsto 0$, we get

$$(x,0) = \sum_{r=0}^{\infty} \frac{(\gamma)_r\, x^r}{r!\ \Gamma(\alpha+r)\Gamma(\beta)} = \frac{1}{\Gamma(\beta)}\ E^{\gamma}_{1,\alpha}(x), \tag{1.12}$$

(ii) For $r \mapsto 0$, we get

$$E^{\gamma}_{\alpha,\beta,k}(0,z) = \sum_{s=0}^{\infty} \frac{(\gamma)_s\, y^{ks}}{\Gamma(\alpha)\Gamma(\beta+ks)s!} = \frac{1}{\Gamma(\alpha)}\ E^{\gamma}_{k,\beta}(y^k), \tag{1.13}$$

We infer from (1.11) and (1.2), (1.3) that

$$E^{\gamma}_{\alpha,\beta,k}(x,y) = \sum_{r=0}^{\infty} E^{\gamma+r}_{k,\beta}(y^k)\ \frac{(\gamma)_r x^r}{\Gamma(\alpha+r)r!}, \tag{1.14}$$

and

$$E^{\gamma}_{\alpha,\beta,k}(x,y) = \sum_{s=0}^{\infty} E^{\gamma}_{1,\alpha}(x)\ \frac{(\gamma+r)_s y^{ks}}{\Gamma(\beta+ks)s!}. \tag{1.15}$$

Formulas (1.14) and (1.15) are very useful in obtaining other needed properties for the function $E^{\gamma}_{\alpha,\beta,k}(x,y)$. Note that the function $E^{\gamma}_{\alpha,\beta}(z)$ is a special case of the Wright generalized hypergeometric function ${}_p\Psi_q$ (see (Gorenflo R., Kilbas A.A., Mainardi F. & Rogosin S., 2020)):

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)}\ {}_1\Psi_1\left[z\Big|{(\gamma,1) \atop (\beta,\alpha)}\right], \tag{1.16}$$

where (see (Srivastava H.M. & Manocha H.L., 1984)):

$${}_p\Psi_q\left[{(\alpha_1,A_1),\dots,(\alpha_p,A_p); \atop (\beta_1,B_1),\dots,(\beta_q,B_q);}\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\alpha_j+A_j n)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_j n)}\frac{z^n}{n!}, \tag{1.17}$$

and the coefficients $A_1,\dots,A_p$ and $B_1,\dots,B_q$ are positive real numbers such that

$$1+\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0.$$

Hence the series expansions (1.14) and (1.15), can be rewritten in the forms

$$E^{\gamma}_{\alpha,\beta,k}(\mathrm{x},\mathrm{y}) = \sum_{r=0}^{\infty}\ {}_1\Psi_1\left[y^k\Big|{(\gamma+\mathrm{r},1) \atop (\beta,k)}\right]\frac{(\gamma)_r x^r}{\Gamma(\gamma+\mathrm{r})\Gamma(\alpha+\mathrm{r})r!}, \tag{1.18}$$

and

$$E^{\gamma}_{\alpha,\beta,k}(x,y) = \sum_{s=0}^{\infty}\ {}_1\Psi_1\left[x\Big|{(\gamma,1) \atop (\alpha,1)}\right]\frac{(\gamma+r)_s y^{ks}}{\Gamma(\beta+ks)s!}, \tag{1.19}$$

respectively.

This paper is rather technical and is devoted to some analytic and computational properties of the Mittag-Leffler function of two variables $E^{\gamma}_{\alpha,\beta,k}(x,y)$. The layout of the paper is as follows. In Section 2, we give several integral representations involving the bivariate Mittag-Leffler function $E^{\gamma}_{\alpha,\beta,k}(x,y)$. In Section 3, we establish differential relations for $E^{\gamma}_{\alpha,\beta,k}(x,y)$. In Section 4, we discuss some useful integral transforms like the Mellin transform, Laplace transform, Euler transform and Whittaker transform.

2. Integral Representations

First of all, we establish an integral representation for $E^{\gamma}_{\alpha,\beta,k}(x,y)$ that is derived directly from Hankle representation of Gamma function

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{c} e^t t^{-z} dt, \tag{2.1}$$

where the path of integration is a simple loop beginning and ending at $-\infty$ and encircling the origin in the positive direction.

Theorem 2.1: *Let* $\{\Re(\beta), \Re(\gamma), \Re(\alpha)\} > 0$, *then*

$$E^{\gamma}_{\alpha,\beta,k}(x,y) = \frac{1}{(2\pi i)^2}\int_C e^{t+\sigma} t^{-\alpha}\sigma^{-\beta}\left(\frac{t\sigma^k}{t(\sigma^k - y^k) - x\sigma^k}\right)^{\gamma} dt. \tag{2.2}$$

Proof: By Letting $z = \alpha + r$ in (2.1), we get

$$\frac{1}{\Gamma(\alpha + r)} = \frac{1}{2\pi i}\int_C e^t t^{-(\alpha+r)}\, dt \tag{2.3}$$

and by Letting $z = \beta + ks$ in (2.1), we get

$$\frac{1}{\Gamma(\beta + ks)} = \frac{1}{2\pi i}\int_C e^{\sigma}\sigma^{-(\beta+ks)}\, d\sigma. \tag{2.4}$$

Hence

$$\frac{1}{\Gamma(\alpha + r)\,\Gamma(\beta + ks)} = \frac{1}{(2\pi i)^2}\int_C\int_C e^{t+\sigma} t^{-(\alpha+r)}\,\sigma^{-(\beta+ks)}\, dt d\sigma,$$

which on multiplying both sides by $\frac{(\gamma)_{r+s}\, x^r y^{ks}}{r!\, s!}$, gives us

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}\, x^r y^{ks}}{\Gamma(\alpha + r)\,\Gamma(\beta + ks)\, r!\;s!} = \frac{1}{(2\pi i)^2}\int_C\int_C e^{t+\sigma} t^{-\alpha}\,\sigma^{-\beta}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}\,\left(\frac{x}{t}\right)^r\left(\frac{y}{\sigma}\right)^{ks}}{r!\;s!}\, dt d\sigma.$$

From $(\gamma)_{r+s} = (\gamma)_r\,(\gamma + r)_s$, we get

$$\begin{aligned}
E^{\gamma}_{\alpha,\beta,k}(x,y) &= \frac{1}{(2\pi i)^2}\int_C\int_C e^{t+\sigma} t^{-\alpha}\,\sigma^{-\beta}\sum_{r=0}^{\infty}\frac{(\gamma)_r\left(\frac{x}{t}\right)^r}{r!}\sum_{s=0}^{\infty}\frac{(\gamma + r)_s\,\left(\frac{y}{\sigma}\right)^{ks}}{s!}\, dt d\sigma \\
&= \frac{1}{(2\pi i)^2}\int_C\int_C e^{t+\sigma} t^{-\alpha}\,\sigma^{-\beta}\left(1 - \frac{y^k}{\sigma^k}\right)^{-(\gamma+r)}\sum_{r=0}^{\infty}\frac{(\gamma)_r\left(\frac{x}{t}\right)^r}{r!}\, dt\, d\sigma \\
&= \frac{1}{(2\pi i)^2}\int_C\int_C e^{t+\sigma} t^{-\alpha}\,\sigma^{-\beta}\left(\frac{\sigma^k}{\sigma^k - y^k}\right)^{\gamma}\sum_{r=0}^{\infty}\frac{(\gamma)_r\left(\frac{x\,\sigma^k}{t(\sigma^k - y^k)}\right)^r}{r!}\, dt\, d\sigma \\
&= \frac{1}{(2\pi i)^2}\int_C\int_C e^{t+\sigma} t^{-\alpha}\,\sigma^{-\beta}\left(\frac{\sigma^k}{\sigma^k - y^k}\right)^{\gamma}\left(1 - \frac{x\,\sigma^k}{t(\sigma^k - y^k)}\right)^{-\gamma} dt d\sigma \\
&= \frac{1}{(2\pi i)^2}\int_C\int_C e^{t+\sigma} t^{-\alpha}\,\sigma^{-\beta}\left(\frac{t\sigma^k}{t(\sigma^k - y^k) - x\,\sigma^k}\right)^{\gamma} dt d\sigma
\end{aligned}$$

which is the desired result (2.2).

Next, we express $E^{\gamma}_{\alpha,\beta,k}(x,y)$ as the Mellin-Barnes type integral.

Theorem 2.2: *Let* $\{\Re(\beta), \Re(\gamma), \Re(\alpha)\} > 0, then$

$$E^{\gamma}_{\alpha,\beta,k}(x,y) = \frac{1}{2\pi i}\frac{1}{\Gamma(\gamma)}\left\{{}_0F_1\,[0;\alpha;x]\right\}\int_{L_s}\frac{\Gamma(s)\,\Gamma(\gamma + r - s)}{\Gamma(\beta - ks)}(-y^k)^{-s} ds, \tag{2.4}$$

where $\{|arg\,(z_1)|, |arg\,(z_2)|\} < \pi$*; and we assume that the contour* L_s *is in the s-plane and runs from* $c - i\infty$ *to* $c + i\infty$ *and the contour* L_t *is in the t-plane and runs from* $C - i\infty$ *to* $c + i\infty$.

Proof: Starting from the assertion (1.14) and replacing $E^{\gamma}_{\alpha,\beta}(z)$ by its Mellin-Barnes integral representation (1.7), we get

$$\begin{aligned}
E^{\gamma}_{\alpha,\beta,k}(x,y) &= \sum_{r=0}^{\infty}\left[\frac{1}{2\pi i}\frac{1}{\Gamma(\gamma + r)}\int_{L_s}\frac{\Gamma(s)\,\Gamma(\gamma + r - s)}{\Gamma(\beta - ks)}(-y^k)^{-s} ds\right]\frac{(\gamma)_r x^r}{\Gamma(\alpha + r) r!}, \\
&= \frac{1}{2\pi i}\frac{1}{\Gamma(\gamma + r)}\frac{(\gamma)_r x^r}{\Gamma(\alpha + r) r!}\int_{L_s}\frac{\Gamma(s)\,\Gamma(\gamma + r - s)}{\Gamma(\beta - ks)}(-y^k)^{-s} ds,
\end{aligned}$$

$$= \frac{1}{2\pi i}\frac{x^r}{\Gamma(\gamma)\Gamma(\alpha+r)r!}\int_{L_s}\frac{\Gamma(s)\,\Gamma(\gamma+r-s)}{\Gamma(\beta-ks)}(-y^k)^{-s}ds,$$

$$= \frac{1}{2\pi i}\frac{1}{\Gamma(\gamma)}\left\{{}_0F_1\,[0;\alpha;x]\right\}\int_{L_s}\frac{\Gamma(s)\,\Gamma(\gamma+r-s)}{\Gamma(\beta-ks)}(-y^k)^{-s}ds$$

$$E^{\gamma}_{\alpha,\beta,k}(x,y) = \frac{1}{2\pi i}\frac{1}{\Gamma(\gamma)}\left\{{}_0F_1\,[0;\alpha;x]\right\}\int_{L_s}\frac{\Gamma(s)\,\Gamma(\gamma+r-s)}{\Gamma(\beta-ks)}(-y^k)^{-s}ds,$$

which is the desired result.

Further, we derive several integral formulas involving $E^{\gamma}_{\alpha,\beta,k}(x,y)$.

Theorem 2.3: *Let* $\{\Re(\beta),\Re(\gamma),\Re(\alpha)\}>0$, *then*

$$E^{\gamma}_{\alpha,\beta+\lambda,k}(x,y) = \frac{1}{\Gamma(\lambda)}\int_0^1 t^{\beta-1}(1-t)^{\lambda-1}E^{\gamma}_{\alpha,\beta+\lambda,k}(x,yt)\;dt, \tag{2.5}$$

$$E^{\gamma}_{\alpha,\beta+\lambda,k}(x,y) = \frac{1}{\Gamma(\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{\beta-1}\,E^{\gamma}_{\alpha,\beta,k}(x,y(1-t))\;dt, \tag{2.6}$$

$$z^{\beta}E^{\gamma}_{\alpha,\beta+1,k}(x,yz) = \int_0^z t^{\beta-1}\;E^{\gamma}_{\alpha,\beta,k}(x,yt)\,dt, \tag{2.7}$$

$$s^{-\beta}\left(\frac{\delta^k}{s^k-y^k}\right)^{\gamma}E^{\gamma}_{1,\alpha}\left(\frac{x\delta^k}{s^k-y^k}\right) = \int_0^{\infty} e^{-st}\,t^{\beta-1}\,E^{\gamma}_{\alpha,\beta,k}(x,yt)\,dt, \tag{2.8}$$

$$(x-t)^{\beta+\delta-1}E^{\gamma}_{\alpha,\beta+\delta,k}\big(x,y(x-t)\big)$$
$$= \frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\beta-1}\;E^{\gamma}_{\alpha,\beta,k}\big(x,y(s-t)\big)\,ds. \tag{2.9}$$

Proof: We have

$$\frac{1}{\Gamma(\lambda)}\int_0^1 t^{\beta-1}(1-t)^{\lambda-1}E^{\gamma}_{\alpha,\beta+\lambda,k}(x,y)\;dt$$

$$= \frac{1}{\Gamma(\lambda)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}\,x^r y^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks)\,r!\,s!}\int_0^1 t^{ks+\beta-1}(1-t)^{\lambda-1}\;dt$$

$$= \frac{1}{\Gamma(\lambda)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma)_{r+s}\,x^r y^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks)\,r!\,s!}\frac{\Gamma(\beta+ks)\Gamma(\lambda)}{\Gamma(ks+\beta+\lambda)}$$

$$= \frac{1}{\Gamma(\lambda)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma)_{r+s}\,x^r y^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+\lambda+ks)\,r!\,s!}$$

$$= E^{\gamma}_{\alpha,\beta+\lambda,k}(x,y)$$

which is the proof of (2.5). We have

$$\frac{1}{\Gamma(\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{\beta-1}\,E^{\gamma}_{\alpha,\beta,k}(x,y(1-t))\;dt$$

$$= \frac{1}{\Gamma(\lambda)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}\,x^r y^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks)\,r!\,s!}\int_0^1 t^{\lambda-1}(1-t)^{ks+\beta-1}\;dt$$

$$= \frac{1}{\Gamma(\lambda)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}\,x^r y^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks)\,r!\,s!}\frac{\Gamma(\lambda)\Gamma(ks+\beta)}{\Gamma(ks+\beta+\lambda)}$$

$$= \frac{1}{\Gamma(\lambda)} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^r y^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+\lambda+ks)\, r!\, s!}$$
$$= E^{\gamma}_{\alpha,\beta+\lambda,k}(x,y)$$

which is the proof of (2.6). We have

$$\int_0^z t^{\beta-1}\, E^{\gamma}_{\alpha,\beta,k}(x,yt)\, dt$$
$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^r y^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks)\, r!\, s!} \int_0^z t^{ks+\beta-1}\, dt$$
$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^r y^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks)\, r!\, s!} \frac{z^{ks+\beta}}{(ks+\beta)}$$
$$= z^{\beta} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^r (yz)^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks+1)\, r!\, s!}$$
$$= z^{\beta} E^{\gamma}_{\alpha,\beta+1\,,k}(x,yz),$$

which is the proof of (2.7). We have

$$\int_0^{\infty} e^{-st} t^{\beta-1}\, E^{\gamma}_{\alpha,\beta\,,k}(x,yt)\, dt$$
$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^r y^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks)\, r!\, s!} \int_0^{\infty} e^{-st} t^{ks+\beta-1}\, dt$$
$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^r y^{ks} \delta^{-(ks+\beta)}}{\Gamma(\alpha+r)\, r!\, s!}$$
$$= s^{-\beta} \sum_{r=0}^{\infty} \frac{(\gamma)_r\, (x)^r}{\Gamma(\alpha+r)\, r!} \sum_{s=0}^{\infty} \frac{(\gamma+r)_s \left(\frac{y}{\delta}\right)^{ks}}{s!},$$
$$= s^{-\beta} \left(1 - \frac{y^k}{\delta^k}\right)^{-(\gamma+r)} \sum_{r=0}^{\infty} \frac{(\gamma)_r\, (x)^r}{\Gamma(\alpha+r)\, r!}$$
$$= s^{-\beta} \left(\frac{\delta^k}{\delta^k - y^k}\right)^{\gamma} \sum_{r=0}^{\infty} \frac{(\gamma)_r \left(\frac{x\delta^k}{\delta^k - y^k}\right)^r}{\Gamma(\alpha+r)\, r!}$$
$$= s^{-\beta} \left(\frac{\delta^k}{s^k - y^k}\right)^{\gamma} E^{\gamma}_{1,\alpha}\left(\frac{x\delta^k}{s^k - y^k}\right)$$

which is the proof of (2.8).

Let $u = \frac{s-t}{x-t}$ in the right-hand side of the assertion (2.9), then

$$\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1} (s-t)^{\beta-1}\, E^{\gamma}_{\alpha,\beta\,,k}(x, y(s-t))\, ds$$
$$= \frac{1}{\Gamma(\delta)} \int_0^1 (x-t)^{\delta-1} (1-u)^{\delta-1} u^{\beta-1} (x-t)^{\beta-1} (x-t) \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^r y^{ks} u^{ks} (x-t)^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks)\, r!\, s!}\, du$$
$$= \frac{1}{\Gamma(\delta)} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^r y^{ks} (x-t)^{\delta+\beta+ks-1}}{\Gamma(\alpha+r)\Gamma(\beta+ks)\, r!\, s!} \times \int_0^1 (1-u)^{\delta-1} u^{ks+\beta-1}\, du$$

$$= \frac{1}{\Gamma(\delta)} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^r y^{ks} (x-t)^{\delta+\beta+ks-1}}{\Gamma(\alpha+r)\Gamma(\beta+ks)\, r!\, s!} \times \frac{\Gamma(\delta)\Gamma(ks+\beta)}{\Gamma(ks+\beta+\delta)}$$

$$= (x-t)^{\beta+\delta-1} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^r (y(x-t))^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks)\, r!\, s!}$$

$$= (x-t)^{\beta+\delta-1} E^{\gamma}_{\alpha,\beta+\delta,k}\big(x, y(x-t)\big),$$

which is the proof of (2.9).

3. Differential Relations

For purpose of this section, we recall the useful fractional derivative formula (see (Podlubny I., 1999)):

$$\widehat{D}_Z^{v}\, z^a = \frac{\Gamma(a+1)}{\Gamma(a-v+1)} z^{a-v}, \tag{3.1}$$

we derive the following interesting differential relations.

Theorem 3.1: *If* $\{\Re(\beta), \Re(\gamma), \Re(\alpha)\} > 0\,,\ p \in \mathbb{N}$, *then*

$$\widehat{D}_x^{p}\left\{E^{\gamma}_{\alpha,\beta,k}(x,y)\, x^{\alpha-1}\right\} = x^{\alpha-p-1}\, E^{\gamma}_{\alpha-p,\beta,k}\,(x,y), \tag{3.2}$$

$$\widehat{D}_y^{p}\left\{E^{\gamma}_{\alpha,\beta,k}(x,y)\, y^{\beta-1}\right\} = y^{\beta-p-1}\, E^{\gamma}_{\alpha,\beta-p,k}\,(x,y), \tag{3.3}$$

$$\frac{\partial^2}{\partial x \partial y}\left\{E^{\gamma}_{\alpha,\beta,k}(x,y)\, x^{\alpha} y^{\beta}\right\} = x^{\alpha-1} y^{\beta-1}\, E^{\gamma}_{\alpha-1,\beta-1,k}\,(x,y). \tag{3.4}$$

Proof: From definitions (1.11) and (3.1), we find that

$$\widehat{D}_x^{p}\left\{E^{\gamma}_{\alpha,\beta,k}(x,y)\, x^{\alpha-1}\right\} = \widehat{D}_x^{p}\left\{\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^{r+\alpha-1} y^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks)\, r!\, s!}\right\}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^{r+\alpha-p-1} y^{ks}}{\Gamma(\alpha-p+r)\Gamma(\beta+ks)\, r!\, s!}$$

$$= x^{\alpha-p-1} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^{r} y^{ks}}{\Gamma(\alpha-p+r)\Gamma(\beta+ks)\, r!\, s!},$$

which is the result (3.2).

$$\widehat{D}_y^{p}\left\{E^{\gamma}_{\alpha,\beta,k}(x,y)\, y^{\beta-1}\right\} = \widehat{D}_y^{p}\left\{\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^{r} y^{ks+\beta-1}}{\Gamma(\alpha+r)\Gamma(\beta+ks)\, r!\, s!}\right\}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^{r} y^{ks+\beta-p-1}}{\Gamma(\alpha+r)\Gamma(\beta-p+ks)\, r!\, s!}$$

$$= y^{\beta-p-1} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^{r} y^{ks}}{\Gamma(\alpha+r)\Gamma(\beta-p+ks)\, r!\, s!},$$

which is the result (3.3).

$$\frac{\partial^2}{\partial x \partial y}\left\{E^{\gamma}_{\alpha,\beta,k}(x,y)\, x^{\alpha} y^{\beta}\right\} = \frac{\partial^2}{\partial x \partial y}\left\{\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^{r+\alpha} y^{ks+\beta}}{\Gamma(\alpha+r)\Gamma(\beta+ks)\, r!\, s!}\right\}$$

$$= \frac{\partial}{\partial y}\left\{\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^{r+\alpha-1} y^{ks+\beta}}{\Gamma(\alpha-1+r)\Gamma(\beta+ks)\, r!\, s!}\right\}$$

$$= x^{\alpha-1} y^{\beta-1} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^{r} y^{ks}}{\Gamma(\alpha-1+r)\Gamma(\beta-1+ks)\, r!\, s!}$$

which is the result (3.4).

4. Integral Transforms

In this section, we establish some useful integral transforms like Euler transform, Laplace transform, Mellin transform and Whittaker transform. First, we aim to obtain Whittaker transform for the function $E^{\gamma}_{\alpha,\beta,k}(x,y)$. The Whittaker function $W_{k,\mu}(z)$ is defined by (see (Srivastava H.M. & Manocha H.L., 1984, p. 39; Shukla A.K. & Prajapati J.C., 2007)):

$$W_{k,\mu}(z) = z^{\mu+\frac{1}{2}} exp\left(-\frac{z}{2}\right)\ {}_1F_1\left(\mu - k + \frac{1}{2}; 2\mu + 1; z\right),$$

where ${}_1F_1$ is Kummer's function (Srivastava H.M. & Manocha H.L., 1984), and possess the following integral formula:

$$\int_0^\infty e^{\frac{-t}{2}} t^{v-1}\, W_{k,\mu}(t)\, dt = \frac{\Gamma\left(\frac{1}{2}+\mu+v\right)\,\Gamma\left(\frac{1}{2}-\mu+v\right)}{\Gamma(1-\lambda+v)},\ \ \Re(v\pm\mu) > -\frac{1}{2}.$$

Theorem 4.1 (Whittaker transform): $If\ \{\Re(\beta),\Re(\gamma),\Re(\alpha)\} > 0, then$

$$\int_0^\infty e^{\frac{-pt}{2}}\, t^{\rho-1} W_{k,\mu}(pt) \times E^{\gamma}_{\alpha,\beta,k}(xt^s, yt^\sigma)\, dt$$

$$= p^{-\rho} \sum_{r=0}^{\infty} {}_3\Psi_2\left[\begin{matrix}(\gamma+r,1),\left(\frac{1}{2}+\delta r+\rho+\mu,\sigma\right),\left(\frac{1}{2}+\delta r+\rho-\mu,\sigma\right); \\ \\ (\beta,1),(1+\delta r+\rho-\lambda,\sigma);\end{matrix}\ \ \frac{y}{p^\sigma}\right] \times \frac{(\gamma)_r\left(\frac{x}{p^\delta}\right)^r}{\Gamma(\alpha+r)\Gamma(\gamma+r)r!}, \tag{4.1}$$

Proof: Let $pt = u$ in the left-hand side of assertion (4.1), we get

$$\frac{1}{p}\int_0^\infty e^{\frac{-u}{2}}\left(\frac{u}{p}\right)^{\rho-1} W_{k,\mu}(u) \times E^{\gamma}_{\alpha,\beta,k}\left(x\left(\frac{u}{p}\right)^s, y\left(\frac{u}{p}\right)^\sigma\right) du$$

$$= p^{-\rho}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}\left(\frac{x}{p^\delta}\right)^r\left(\frac{y}{p^\sigma}\right)^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks)r!\,s!} \times \int_0^\infty e^{\frac{-u}{2}} u^{\delta r+\sigma ks+\rho-1}\, W_{k,\mu}(u)\, du$$

$$= p^{-\rho}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}\left(\frac{x}{p^\delta}\right)^r\left(\frac{y}{p^\sigma}\right)^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks)r!\,s!} \times \frac{\Gamma\left(\frac{1}{2}+\delta r+\sigma ks+\rho+\mu\right)\Gamma\left(\frac{1}{2}+\delta r+\sigma ks+\rho-\mu\right)}{\Gamma(1+\delta r+\sigma ks+\rho-\lambda)}$$

$$= p^{-\rho}\sum_{r=0}^{\infty} {}_3\Psi_2\left[\begin{matrix}(\gamma+r,1),\left(\frac{1}{2}+\delta r+\rho+\mu,\sigma k\right),\left(\frac{1}{2}+\delta r+\rho-\mu,\sigma k\right); \\ \\ (\beta,k),(1+\delta r+\rho-\lambda,\sigma k);\end{matrix}\ \ \frac{y^k}{p^\sigma}\right] \times \frac{(\gamma)_r\left(\frac{x}{p^\delta}\right)^r}{\Gamma(\alpha+r)\Gamma(\gamma+r)r!}$$

which in view of (1.18), yields the assertion (4.1).

Theorem 4.2 (Euler transform)**:** $If\ \{\Re(\beta),\Re(\gamma),\Re(\alpha)\} > 0, then$

$$\int_0^1\int_0^1 z_1^{a_1-1}(1-z_1)^{b_1-1}z_2^{a_2-1}(1-z_2)^{b_2-1}E^{\gamma}_{\alpha,\beta,k}\left(xz_1^\sigma, yz_2^\delta\right)dz_1dz_2$$

$$= \Gamma(b_1)\Gamma(b_2)\sum_{r=0}^{\infty} {}_2\Psi_2\left[\begin{matrix}(\gamma+r,1)(a_2,\delta k); \\ \\ (\beta,k),(a_2+b_2,\delta k);\end{matrix}\ \ y^k\right] \times \frac{(\gamma)_r\,\Gamma(a_1+\sigma r)\,x^r}{r!\,\Gamma(\alpha+r)\Gamma(\gamma+r)\Gamma(a_1+b_1+\sigma r)}. \tag{4.2}$$

Proof: Denote the left-hand side of equation (4.2) by I, then from the definition (1.11), we get

$$I = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}\,x^r y^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks)r!\,s!} \times \int_0^\infty z_1^{\sigma r+a_1-1}(1-z_1)^{b_1-1}\,dz_1 \times \int_0^\infty z_2^{\delta ks+a_2-1}(1-z_2)^{b_2-1}\,dz_2.$$

Now, by using the Beta function (see e.g., (Prabhakar T.R., 1971)):

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

we obtain

$$I = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma)_{r+s}\,\Gamma(\sigma m+a_1)\Gamma(b_1)\Gamma(\delta n+a_2)\Gamma(b_2)\,x^r y^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks)r!\,s!\,\Gamma(\sigma m+a_1+b_1)\Gamma(\delta n+a_2+b_2)},$$

which because of (1.18), yields the assertion (4.2).

Theorem 4.3 (Laplace transform)**:** $If\ \{\Re(\beta), \Re(\gamma), \Re(\alpha)\} > 0, then$

$$\int_0^\infty \int_0^\infty z_1^{a_1-1} e^{-s_1 z_1} z_2^{a_2-1} e^{-s_2 z_2} E_{\alpha,\beta,k}^{\gamma}\left(x z_1^{\sigma_1}, y z_2^{\sigma_2}\right) dz_1 dz_2$$

$$= s_1^{-a_1} s_2^{-a_2} \sum_{m=0}^{\infty} {}_2\Psi_1 \left[\begin{matrix} (\gamma+r,1)(a_2,\sigma_2 k); & \\ & y^k s_2^{-\sigma_2 k} \\ (\beta,k); & \end{matrix}\right] \times \frac{(\gamma)_r \Gamma(a_1+\sigma_1 r)\, x^r\, s_1^{-\sigma_1 r}}{\Gamma(\gamma+r)\Gamma(\alpha+r)\, r!}, \tag{4.3}$$

Proof: Denote the left-hand side of equation (4.3) by me, then from the definition (1.11), we get

$$I = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}\, x^r y^{ks}}{\Gamma(\alpha+r)\Gamma(\beta+ks) r!\, s!} \times \int_0^\infty z_1^{\sigma_1 r + a_1 - 1} e^{-s_1 z_1}\, dz_1$$
$$\times \int_0^\infty z_2^{\sigma_2 ks + a_2 - 1} e^{-s_2 z_2}\ dz_2. \tag{4.4}$$

Now, by using the integral representation of the Gamma function (Rainville E.D., 1960)

$$a^{-z}\Gamma(z) = \int_0^\infty t^{z-1} e^{-at} dt,$$

we obtain

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\gamma)_{r+s}\, \Gamma(\sigma_1 r + a_1)\Gamma(\sigma_2 ks + a_2)\, x^r y^{ks}\, s_1^{-(\sigma_1 r + a_1)} s_2^{-(\sigma_2 ks + a_2)}}{\Gamma(\alpha+r)\Gamma(\beta+ks) r!\, s!}$$

$$= s_1^{-a_1} s_2^{-a_2} \sum_{m=0}^{\infty} {}_2\Psi_1 \left[\begin{matrix} (\gamma+r,1)(a_2,\sigma_2 k); & \\ & y^k s_2^{-\sigma_2 k} \\ (\beta,k); & \end{matrix}\right] \times \frac{(\gamma)_r \Gamma(a_1+\sigma_1 r)\, x^r\, s_1^{-\sigma_1 r}}{\Gamma(\gamma+r)\Gamma(\alpha+r)\, r!}$$

which because of (1.18), yields the assertion (4.3).

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