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Compact High Order Finite Volume Method and Its Application

Xiang Li[1]

[1] Jinan University, Guangdong 511400, China

Correspondence: Xiang Li, Jinan University, Guangdong 511400, China.

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Abstract

This paper presents a compact high-order finite volume method based on recording the weighted integral value of a function on a grid cell. Extending the general finite volume method, it employs a single unit template and a limiter method to suppress oscillations, addressing the convection-diffusion equation. By utilizing implicit methods for nonlinear and linear solvers of the discrete equation, the scheme achieves enhanced compactness, accuracy, and computational efficiency.

Keywords: numerical format, diffusion equation, compact finite volume method

1. Introduction

1.1 Diffusion Equation

An equation with the partial derivative of an unknown function $u(x_1, x_2, \ldots, x_n, t)$ is called a partial differential equation u is a function of n+1 independent variables, t is the time variable, $x_1, x_2, \ldots, x_n$ represents the spatial variable.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left(k_1 \frac{\partial u}{\partial x_1}\right) + \cdots + \frac{\partial}{\partial x_n}\left(k_n \frac{\partial u}{\partial x_n}\right) + F(\boldsymbol{x}, t)$$

Where, $\boldsymbol{u} = u(\boldsymbol{x}, t)$ is the concentration of a substance in the diffusion process, or the temperature at x and time t in the heat transfer process of a solid. Coefficient $k_i = k_i(\boldsymbol{x}) > 0\ (i = 1, \ldots, n)$ is called the diffusion coefficient or thermal conductivity coefficient. When $k_1 = k_2 = \cdots = k_n = a\ (a > 0)$, the equation is:

$$\frac{\partial u}{\partial t} = a\Delta \boldsymbol{u} + F(\boldsymbol{x}, t)$$

When n = 1 first-order diffusion (heat transfer) equation is obtained: $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + F(\boldsymbol{x}, t)$.

In the case n=2: $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = k\Delta u + F$ is the convective diffusion equation on the plane, where $u = u(x, y, t)$ represents the concentration of a substance in the flow field, $a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}$ is the convective term, where (a, b) is the flow rate, $k\Delta u$ is the diffusion term, k is the diffusion coefficient, k>0.

1.2 Compact Finite Volume Method

The numerical solution of partial differential equations (PDEs) is crucial in computational mathematics. High-precision schemes include the spectral method, spectral element method, compact difference scheme, weighted essentially non-oscillatory (WENO) scheme, discontinuous Galerkin finite element method, and various hybrid algorithms. This section focuses on the finite volume method (FVM), also known as the control volume method, which is based on the conservation of physical quantities. The computational domain is divided into primary and dual sections, forming non-overlapping control volumes around each grid point. Integrating the differential equations over each control volume results in a set of discrete equations with the unknowns being the values of the dependent variables at the grid points.

FVM ensures local conservation but often complicates convergence analysis due to phase error considerations. The compact finite volume method, a high-precision scheme derived from discretizing equations using the finite volume approach based on difference schemes, combines the advantages of FVM with ease of solution.

2. Results and Discussions

2.1 Construction of Compact High Order Finite Volume Schemes

Paper (Xuan LJ & Wu JZ, 2010) presents a high-order numerical scheme for solving hyperbolic conservation law equations based on weighted integrals. We will draw on the idea of recording the weighted integral value of the function on the grid cell to give a compact high order finite volume method. The differences in the papers (Xuan LJ & Wu JZ, 2010) are as follows:

(1) The multi-unit template is used in the papers (Xuan LJ & Wu JZ, 2010) to construct the high-order format, and we will use the template of a single unit. In this case, the recorded information of each unit needs to be increased correspondingly when constructing the high-order format. The advantages are that the format has better compactness, convenient parallel computation and easy to maintain the precision at the boundary;

(2) Paper (Xuan LJ & Wu JZ, 2010) uses the idea of weighted intrinsically non-oscillatory scheme (WENO) (Liu XD, Osher S & Chan T, 1994) for reference to suppress the oscillation of numerical solutions at discontinuities. This processing method requires a large amount of calculation, so we will adopt the method of limiter to suppress the oscillation and consider a new precision preserving limiter WBAP limiter (Li W, Ren YX, Lei G & Luo H, 2011; Li W & Ren YX, 2012).

(3) Paper (Xuan LJ & Wu JZ, 2010) only considers the solution of the one-dimensional hyperbolic conservation law equation and adopts the third-order TVD Runge-Kutta scheme for the time direction. We will further consider the convection-diffusion equation. Due to the existence of the diffusion term, the stability of the explicit scheme requires $\Delta t \sim (\Delta x)^2$, which brings great limitations on the time step. We will consider implicit methods. For implicit methods, nonlinear and linear solvers for discrete equations need to be studied.

Next, we give the construction process of compact high order finite volume method.

The one-dimensional convection-diffusion equation in the form of conservation law is considered.

$$u_t + f(u)_x = Du_{xx}, \quad x \in [a, b], \quad t > 0, \tag{1}$$

Where f(u) is the flux function, $D > 0$ is the diffusion coefficient, and $u(x, t)$ is the function to be solved. When $f(u) = au$, is the linear convection equation, where a is the convection velocity. When $f(u) = u^2$, the equation is Burgers equation with diffusion term.

As shown in Figure 1, the interval [a,b] is first divided into N parts:

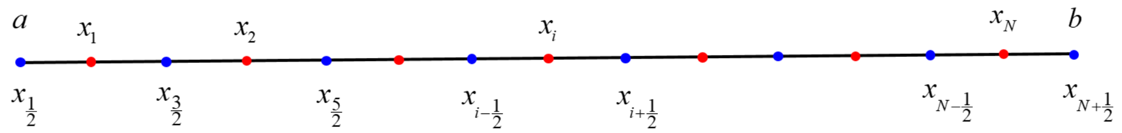


Figure 1. Mesh generation

Define grid cell: $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, then cell center is $x_i = \frac{1}{2}(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, grid size is $\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$. For simplicity, let's use a uniform grid, i.e. $\Delta x_i = \frac{b-a}{N},\ i = 1, 2, \cdots, N$.

On each cell, we record the integration of the function $u(x,t)$ to be solved with a set of weight functions:

$$v_i^k = \int_{I_i} u(x,t)\phi_i^k(x)dx \equiv \left\langle u, \phi_i^k \right\rangle_{I_i} \tag{2}$$

Where, $\left\{\phi_i^k(x)\right\}, k = 0, 1, \cdots, K$ is a given set of linearly independent test functions, and the notation $\langle \cdot \rangle_{I_i}$ represents the integral on the cell I_i. Taking equation (1)× ϕ_i^k and integrating it over the unit gives

$$\int_{I_i} \frac{\partial u}{\partial t}\phi_i^k(x)dx + \int_{I_i} f(u)_x \phi_i^k(x)dx = D\int_{I_i} \frac{\partial^2 u}{\partial x^2}\phi_i^k(x)dx \tag{3}$$

Integration by parts gives us the following weak-form equation

$$\frac{dv_i^k}{dt} + f(u)\phi_i^k\Big|_{\partial I_i} - \int_{I_i} f(u)\frac{\partial \phi_i^k}{\partial x}dx = D\frac{\partial u}{\partial x}\phi_i^k\Bigg|_{\partial I_i} - D\int_{I_i} \frac{\partial u}{\partial x}\frac{\partial \phi_i^k}{\partial x}dx \tag{4}$$

In order to solve equation (4), we need to know the values of u and $\frac{\partial u}{\partial x}$ at the cell boundary $x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}$, and we also need to compute the integral on both cells. The basic idea is to use the v_i^k of unit I_i to reconstruct the polynomial expression of u on unit I_i, and then calculate the required result. The time direction was solved by the third-order TVD Runge-Kutta method.

For ordinary differential equations, the third-order TVD Runge-Kutta method is:

$$u^{(1)} = u^n + \Delta t L(u^n)$$
$$u^{(2)} = \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)})$$
$$u^{n+1} = \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)})$$

Where Δt is the time step and u^n represents the value u at the moment $n\Delta t$.

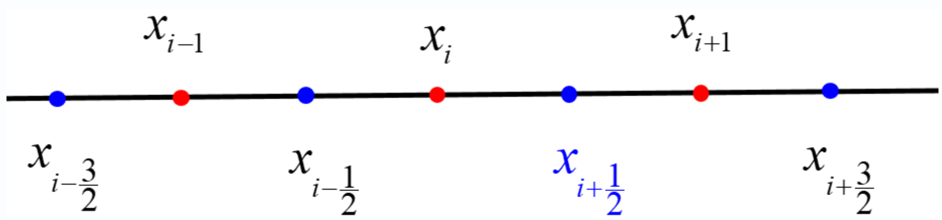


Figure 2.

As shown in Figure 2, for the function values of u and $\frac{\partial u}{\partial x}$ on $x_{i+\frac{1}{2}}$, we can use $v_i^k(k = 0, \cdots, K-1)$ to reconstruct an approximate polynomial of u by interpolation. According to the need of calculation accuracy, different K can be used to construct different order of u interpolation polynomials, so as to get different precision formats. Generally speaking, when the reconstructed polynomial of u is p-1, the numerical scheme constructed is p-order method.

Below we derive the construction of second -, third -, fourth - and sixth-order numerical schemes. Define local coordinate $\xi = \frac{x-x_i}{\Delta x}$, thus

$$v_i^k = \int_{I_i} u(x)\phi_i^k(x)dx = \Delta x \int_{-\frac{1}{2}}^{\frac{1}{2}} u(\xi)\phi_i^k(\xi)d\xi ,$$

Equation (4) turns to:

$$\frac{dv_i^k}{dt} + f(u)\phi_i^k\Big|_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} f(u(\xi))\frac{\partial \phi_i^k}{\partial \xi}d\xi = D\frac{1}{\Delta x}\frac{\partial u}{\partial \xi}\phi_i^k\Big|_{-\frac{1}{2}}^{\frac{1}{2}} - D\frac{1}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial u}{\partial \xi}\frac{\partial \phi_i^k}{\partial \xi}d\xi \tag{5}$$

The test function is selected as the simplest non-dimensional polynomial form:

$$\phi_i^0 = \frac{1}{\Delta x}, \quad \phi_i^1 = \frac{x-x_i}{(\Delta x)^2} = \frac{1}{\Delta x}\xi, \quad \phi_i^2 = \frac{(x-x_i)^2}{(\Delta x)^3} = \frac{1}{\Delta x}\xi^2,$$

$$\phi_i^3 = \frac{(x-x_i)^3}{(\Delta x)^4} = \frac{1}{\Delta x}\xi^3, \quad \phi_i^4 = \frac{(x-x_i)^4}{(\Delta x)^5} = \frac{1}{\Delta x}\xi^4, \cdots$$

Accordingly, equation (5) turns to:

$$\frac{dv_i^0}{dt} + \frac{\hat{f}_{i+1/2} - \hat{f}_{i-1/2}}{\Delta x} = D\frac{\hat{h}_{i+1/2} - \hat{h}_{i-1/2}}{\left(\Delta x\right)^2}, \tag{6}$$

$$\frac{dv_i^1}{dt} + \frac{\hat{f}_{i+1/2} + \hat{f}_{i-1/2}}{2\Delta x} - \frac{1}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(u(\xi))d\xi = D\frac{\hat{h}_{i+1/2} + \hat{h}_{i-1/2}}{2\left(\Delta x\right)^2} - D\frac{1}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial u}{\partial \xi}d\xi, \tag{7}$$

$$\frac{dv_i^2}{dt} + \frac{\hat{f}_{i+1/2} - \hat{f}_{i-1/2}}{4\Delta x} - \frac{2}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(u(\xi))\xi d\xi = D\frac{\hat{h}_{i+1/2} - \hat{h}_{i-1/2}}{4\left(\Delta x\right)^2} - D\frac{2}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial u}{\partial \xi}\xi d\xi, \tag{8}$$

$$\frac{dv_i^3}{dt} + \frac{\hat{f}_{i+1/2} + \hat{f}_{i-1/2}}{8\Delta x} - \frac{3}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(u(\xi))\xi^2 d\xi = D\frac{\hat{h}_{i+1/2} + \hat{h}_{i-1/2}}{8\left(\Delta x\right)^2} - D\frac{3}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial u}{\partial \xi}\xi^2 d\xi, \tag{9}$$

$$\frac{dv_i^4}{dt} + \frac{\hat{f}_{i+1/2} - \hat{f}_{i-1/2}}{16\Delta x} - \frac{4}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(u(\xi))\xi^3 d\xi = D\frac{\hat{h}_{i+1/2} - \hat{h}_{i-1/2}}{16\left(\Delta x\right)^2} - D\frac{4}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial u}{\partial \xi}\xi^3 d\xi, \tag{10}$$

$$\cdots$$

$$\frac{dv_i^k}{dt} + \frac{\hat{f}_{i+1/2} - (-1)^k\hat{f}_{i-1/2}}{2^k\Delta x} - \frac{k}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(u(\xi))\xi^{k-1}d\xi = D\frac{\hat{h}_{i+1/2} - (-1)^k\hat{h}_{i-1/2}}{2^k\left(\Delta x\right)^2} - D\frac{k}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial u}{\partial \xi}\xi^{k-1}d\xi \tag{11}$$

Note:

1) Since $v_i^0 = \int_{I_i} u(x)\phi_i^0(x)dx = \frac{1}{\Delta x}\int_{I_i} u(x)dx = \bar{u}_i$, when only v_i^0, which is the average value $\bar{u}_i$ of u on the cell is recorded, the method is reduced to a standard finite volume scheme, i.e., Equation (6).

2) At $x_{i+\frac{1}{2}}$, the left-value function $u_{i+1/2}^L = u_i(\xi = \frac{1}{2})$ is usually different from the right-value function $u_{i+1/2}^R = u_{i+1}(\xi = -\frac{1}{2})$. In order to maintain the conservation of numerical formats. The numerical flux at $x_{i+\frac{1}{2}}$ is calculated by using the Lax-Friedrichs flux function.

$$\hat{f}_{i+1/2} = \hat{f}(u_{i+1/2}^L, u_{i+1/2}^R) \tag{12}$$

Here $\hat{f}(a,b) = \frac{1}{2}[f(a) + f(b) - \delta(b-a)]$, where $\delta = \max_u |f'(u)|$.

When $f(u) = cu$, $\hat{f}_{i+1/2} = \begin{cases} cu_{i+1/2}^L, & c > 0 \\ cu_{i+1/2}^R, & c < 0 \end{cases}$ is the exact windward flux.

3) At $x_{i+\frac{1}{2}}$, the left-value and the right-value of the derivative $(\frac{\partial u}{\partial x})_{i+1/2}$ are also different. We need to figure out the diffusion flux at $x_{i+\frac{1}{2}}$ according to $(\frac{\partial u}{\partial x})_{i+1/2}^L$ and $(\frac{\partial u}{\partial x})_{i+1/2}^R$, which is $\hat{h}_{i+1/2} = h\left((\frac{\partial u}{\partial x})_{i+1/2}^L, (\frac{\partial u}{\partial x})_{i+1/2}^R\right)$. The calculation of the diffusion flux with respect to derivatives is still a developing and improving technique. Depends on the exact solution of diffusive generalized Riemann problem (dGRP) of one-dimensional heat conduction equation, paper (Gassner G, Lörcher F & Munz CD, 2007) derive a diffusion flux calculation method that suitable for finite volume and discontinuous Galerkin (DG) method. In paper (Wang Q, Ren YX, Pan J & Li W, 2017), it is further extended to the finite volume method of Navier-Stokes equations to solve the viscous flux. Following the ideas in paper (Gassner G, Lörcher F & Munz CD, 2007; Wang Q, Ren YX, Pan J & Li W, 2017), the following formula is used to calculate the diffusion flux:

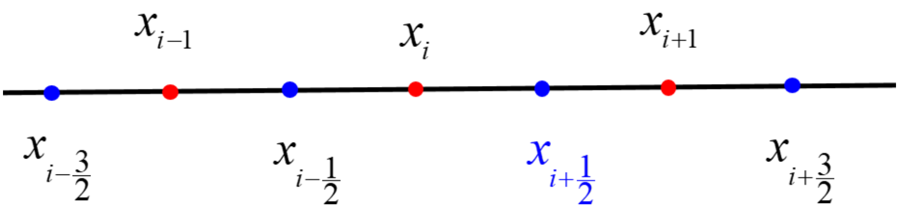
$$\hat{h}_{i+1/2} = \frac{1}{2}\left[(\frac{\partial u}{\partial \xi})_{i+1/2}^L + (\frac{\partial u}{\partial \xi})_{i+1/2}^R\right] + \frac{1}{2\Delta\tilde{x}}(u_{i+1/2}^R - u_{i+1/2}^L), \tag{13}$$

Where $\Delta\tilde{x} = \max\{\Delta x_L, \Delta x_R\}$.

The above is the construction process of the whole numerical scheme, it can be seen that it is an extension of the general finite volume method, we call it compact high order finite volume method. Next, we give the construction process of specific numerical schemes of different orders and the formula of flux calculation.

(1) Construction of second order scheme

Each cell records two values v_i^0, v_i^1. Assuming $u(\xi)=c_0+c_1\xi$, using v_i^0, v_i^1 to reconstruct undetermined coefficients c_0, c_1.



Defined by v_i^0, v_i^1, have

$$v_i^0=\int_{I_i} u(x)\frac{1}{\Delta x}dx=\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\xi)d\xi=\int_{-\frac{1}{2}}^{\frac{1}{2}} c_0+c_1\xi d\xi=c_0$$

$$v_i^1=\int_{I_i} u(x)\frac{\xi}{\Delta x}dx=\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\xi)\xi d\xi=\int_{-\frac{1}{2}}^{\frac{1}{2}} c_0\xi+c_1\xi^2 d\xi=\frac{1}{12}c_1$$

Thus, the reconstruction polynomial on element *i* can be written as:

$$u_i(\xi)=v_i^0+12v_i^1\xi\,. \tag{14}$$

Therefore, there has:

$$u_{i+1/2}^L=u_i(\tfrac{1}{2})=v_i^0+6v_i^1,\quad u_{i-1/2}^R=u_i(-\tfrac{1}{2})=v_i^0-6v_i^1,$$
$$(\frac{\partial u}{\partial \xi})_{i+1/2}^L=(\frac{\partial u}{\partial \xi})_i(\tfrac{1}{2})=12v_i^1,\quad (\frac{\partial u}{\partial \xi})_{i-1/2}^R=(\frac{\partial u}{\partial \xi})_i(-\tfrac{1}{2})=12v_i^1,$$

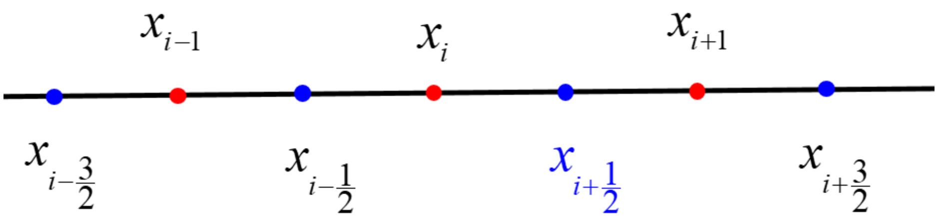
By substituting in equation (13), (14), numerical flux $\widehat{f}_{i+1/2}$ and diffusion flux $\widehat{h}_{i+1/2}$ can be calculated respectively. For the volume fraction in equation (7), by calculating:

$$\frac{1}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\xi)d\xi=\frac{1}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}}(v_i^0+12v_i^1\xi)d\xi=\frac{1}{\Delta x}v_i^0$$
$$\frac{1}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial u}{\partial \xi}d\xi=\frac{1}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}12v_i^1 d\xi=\frac{12v_i^1}{(\Delta x)^2}$$

Substituting into equation (6), (7), gives a second-order format.

(2) Construction of third-order scheme

Each cell records three values v_i^0, v_i^1, v_i^2, assuming $u(\xi)=c_0+c_1\xi+c_2\xi^2, \xi=\frac{x-x_i}{\Delta x}$, using v_i^0, v_i^1, v_i^2 to reconstruct undetermined coefficients c_0, c_1, c_2.



Defined by v_i^0, v_i^1, v_i^2, have

$$v_i^0=\int_{I_i}u(x)\frac{1}{\Delta x}dx=\int_{-\frac{1}{2}}^{\frac{1}{2}}u(\xi)d\xi=\int_{-\frac{1}{2}}^{\frac{1}{2}}c_0+c_1\xi+c_2\xi^2d\xi=c_0+\frac{1}{12}c_2$$

$$v_i^1=\int_{I_i}u(x)\frac{1}{\Delta x}\xi dx=\int_{-\frac{1}{2}}^{\frac{1}{2}}u(\xi)\xi d\xi=\int_{-\frac{1}{2}}^{\frac{1}{2}}c_0\xi+c_1\xi^2+c_2\xi^3d\xi=\frac{1}{12}c_1$$

$$v_i^2=\int_{I_i}u(x)\frac{1}{\Delta x}\xi^2dx=\int_{-\frac{1}{2}}^{\frac{1}{2}}u(\xi)\xi^2d\xi=\int_{-\frac{1}{2}}^{\frac{1}{2}}c_0\xi^2+c_1\xi^3+c_2\xi^4d\xi=\frac{1}{12}c_0+\frac{1}{80}c_2$$

After solving, it obtains:

$$c_0=\frac{9}{4}v_i^0-15v_i^2,$$
$$c_1=12v_i^1,$$
$$c_2=-15v_i^0+180v_i^2$$

Thus, the reconstruction polynomial on element *i* can be written as:

$$u_i(\xi)=\frac{9}{4}v_i^0-15v_i^2+12v_i^1\xi+(-15v_i^0+180v_i^2)\xi^2$$

Therefore, there has:

$$u_{i+1/2}^L=u_i(\tfrac{1}{2})=-\frac{3}{2}v_i^0+6v_i^1+30v_i^2,$$
$$u_{i-1/2}^R=u_i(-\tfrac{1}{2})=-\frac{3}{2}v_i^0-6v_i^1+30v_i^2,$$
$$(\frac{\partial u}{\partial\xi})_{i+1/2}^L=(\frac{\partial u}{\partial\xi})_i(\tfrac{1}{2})=-15v_i^0+12v_i^1+180v_i^2,$$
$$(\frac{\partial u}{\partial\xi})_{i-1/2}^R=(\frac{\partial u}{\partial\xi})_i(-\tfrac{1}{2})=15v_i^0+12v_i^1-180v_i^2,$$

By substituting in equation (13), (14), numerical flux $\widehat{f}_{i+1/2}$ and diffusion flux $\widehat{h}_{i+1/2}$ can be calculated respectively. For the volume fraction in equation (7), (8), by calculating:

$$\frac{1}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}}u(\xi)d\xi=\frac{1}{\Delta x}v_i^0$$

$$\frac{1}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial u}{\partial\xi}d\xi=\frac{12v_i^1}{(\Delta x)^2}$$

$$\frac{2}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}}u(\xi)\xi d\xi=\frac{2}{\Delta x}v_i^1$$

$$\frac{2}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial u}{\partial\xi}\xi d\xi=\frac{1}{(\Delta x)^2}(-5v_i^0+60v_i^2)$$

Substituting into equation (6), (7), (8), gives a third-order format.

(3) The construction of the fourth-order scheme

Each cell records three values $v_i^0, v_i^1, v_i^2, v_i^3$, and the reconstruction polynomial on cell *i* can be obtained:

$$u_i(\xi) = \frac{9}{4}v_i^0 - 15v_i^2 + (75v_i^1 - 420v_i^3)\xi + (-15v_i^0 + 180v_i^2)\xi^2 + (-420v_i^1 + 2800v_i^3)\xi^3$$

$$u_{i+1/2}^L = u_i(\tfrac{1}{2}) = -\frac{3}{2}v_i^0 - 15v_i^1 + 30v_i^2 + 140v_i^3,$$

$$u_{i-1/2}^R = u_i(-\tfrac{1}{2}) = -\frac{3}{2}v_i^0 + 15v_i^1 + 30v_i^2 - 140v_i^3,$$

$$(\frac{\partial u}{\partial \xi})_{i+1/2}^L = (\frac{\partial u}{\partial \xi})_i(\tfrac{1}{2}) = -15v_i^0 - 240v_i^1 + 180v_i^2 + 1680v_i^3,$$

$$(\frac{\partial u}{\partial \xi})_{i-1/2}^R = (\frac{\partial u}{\partial \xi})_i(-\tfrac{1}{2}) = 15v_i^0 - 240v_i^1 - 180v_i^2 + 1680v_i^3,$$

By substituting in equation (13), (14), numerical flux $\widehat{f}_{i+1/2}$ and diffusion flux $\widehat{h}_{i+1/2}$ can be calculated respectively. For the volume fraction in equation (7), (8), (9) by calculating:

$$\frac{1}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\xi)d\xi = \frac{1}{\Delta x}v_i^0$$

$$\frac{1}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial u}{\partial \xi}d\xi = \frac{-30v_i^1 + 280v_i^3}{(\Delta x)^2}$$

$$\frac{2}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\xi)\xi d\xi = \frac{2}{\Delta x}v_i^1$$

$$\frac{2}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial u}{\partial \xi}\xi d\xi = \frac{1}{(\Delta x)^2}(-5v_i^0 + 60v_i^2)$$

$$\frac{3}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\xi)\xi^2 d\xi = \frac{3}{\Delta x}v_i^2$$

$$\frac{3}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial u}{\partial \xi}\xi^2 d\xi = \frac{3}{(\Delta x)^2}(-\frac{19}{2}v_i^1 + 70v_i^3)$$

Substituting into equation (6), (7), (8), (9), gives a fourth-order format.

(4) The construction of the sixth-order scheme

Each cell records three values $v_i^0, v_i^1, v_i^2, v_i^3, v_i^4, v_i^5$, and the reconstruction polynomial on cell *i* can be obtained:

$$u_i(\xi) = \frac{225}{64}v_i^0 - \frac{525}{8}v_i^2 + \frac{945}{4}v_i^4 + (\frac{3675}{16}v_i^1 - \frac{6615}{2}v_i^3 + 10395v_i^5)\xi +$$
$$(-\frac{525}{8}v_i^0 + 2205v_i^2 - 9450v_i^4)\xi^2 + (-\frac{6615}{2}v_i^1 + 56700v_i^3 - 194040v_i^5)\xi^3 +$$
$$(\frac{945}{4}v_i^0 - 9450v_i^2 + 44100v_i^4)\xi^4 + (10395v_i^1 - 194040v_i^3 + 698544v_i^5)\xi^5$$

$$u_{i+1/2}^{L}=u_i(\tfrac{1}{2})=\frac{15}{8}v_i^0+\frac{105}{4}v_i^1-105v_i^2-630v_i^3+630v_i^4+2772v_i^5,$$
$$u_{i-1/2}^{R}=u_i(-\tfrac{1}{2})=\frac{15}{8}v_i^0-\frac{105}{4}v_i^1-105v_i^2+630v_i^3+630v_i^4-2772v_i^5,$$
$$(\frac{\partial u}{\partial \xi})_{i+1/2}^{L}=(\frac{\partial u}{\partial \xi})_i(\tfrac{1}{2})=\frac{105}{2}v_i^0+\frac{1995}{2}v_i^1-2520v_i^2-21420v_i^3+12600v_i^4+83160v_i^5,$$
$$(\frac{\partial u}{\partial \xi})_{i-1/2}^{R}=(\frac{\partial u}{\partial \xi})_i(-\tfrac{1}{2})=-\frac{105}{2}v_i^0+\frac{1995}{2}v_i^1+2520v_i^2-21420v_i^3-12600v_i^4+83160v_i^5,$$

By substituting in equation (13), (14), numerical flux $\widehat{f}_{i+1/2}$ and diffusion flux $\widehat{h}_{i+1/2}$ can be calculated respectively. For the volume fraction in equation (7), (8), (9), (10), by calculating:

$$\frac{1}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}}u(\xi)d\xi=\frac{1}{\Delta x}v_i^0$$
$$\frac{1}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial u}{\partial \xi}d\xi=\frac{1}{(\Delta x)^2}(\frac{105}{2}v_i^1-1260v_i^3+5544v_i^5)$$

$$\frac{2}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}}u(\xi)\xi d\xi=\frac{2}{\Delta x}v_i^1$$
$$\frac{2}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial u}{\partial \xi}\xi d\xi=\frac{2}{(\Delta x)^2}(\frac{7}{8}v_i^0-105v_i^2+630v_i^4)$$

$$\frac{3}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}}u(\xi)\xi^2 d\xi=\frac{3}{\Delta x}v_i^2$$
$$\frac{3}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial u}{\partial \xi}\xi^2 d\xi=\frac{3}{(\Delta x)^2}(\frac{89}{8}v_i^1-315v_i^3+1386v_i^5)$$

$$\frac{4}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}}u(\xi)\xi^3 d\xi=\frac{4}{\Delta x}v_i^3$$
$$\frac{4}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial u}{\partial \xi}\xi^3 d\xi=\frac{4}{(\Delta x)^2}(\frac{15}{32}v_i^0-\frac{117}{4}v_i^2+\frac{315}{2}v_i^4)$$

$$\frac{5}{\Delta x}\int_{-\frac{1}{2}}^{\frac{1}{2}}u(\xi)\xi^4 d\xi=\frac{5}{\Delta x}v_i^4$$
$$\frac{5}{(\Delta x)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial u}{\partial \xi}\xi^4 d\xi=\frac{5}{(\Delta x)^2}(\frac{105}{32}v_i^1-\frac{331}{4}v_i^3+\frac{693}{2}v_i^5)$$

Substituting into equation (6), (7), (8), (9), (10) gives a sixth-order format.

2.2 Numerical Example

In this section, we will list some numerical examples to verify the validity of the compact higher-order finite volume scheme and prove the correctness of the theoretical analysis.

Example 1: linear advection equation

Consider a linear advection equation:

$u_t + u_x = 0$

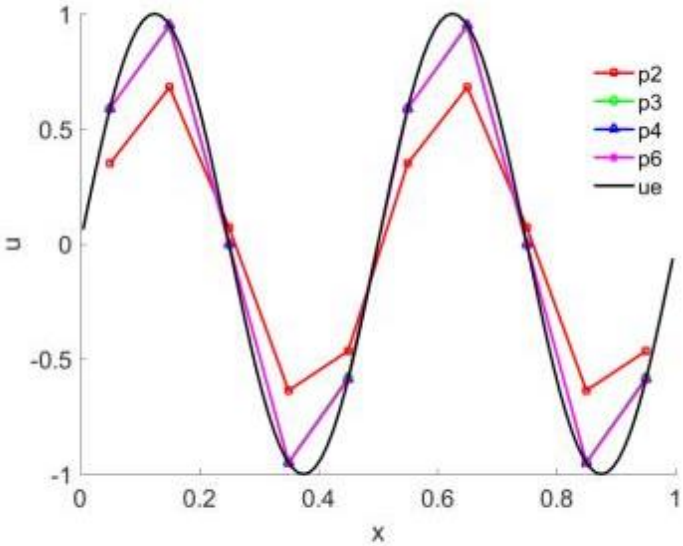
$0 \le x \le 1$, periodic boundary condition

$u_0(x) = \sin(4\pi x)$

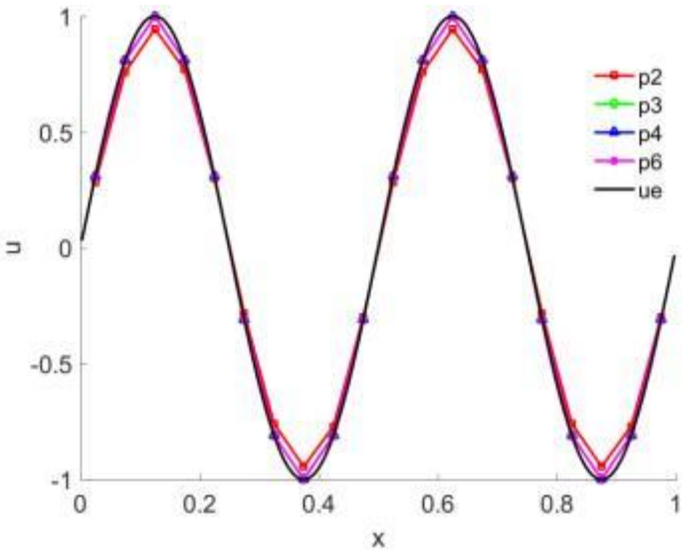
exact solution $u(x) = u_0(x - t)$

time integration TVD RK3, $t_f = 1.0$, $u_0(x) = \sin(4\pi x)$

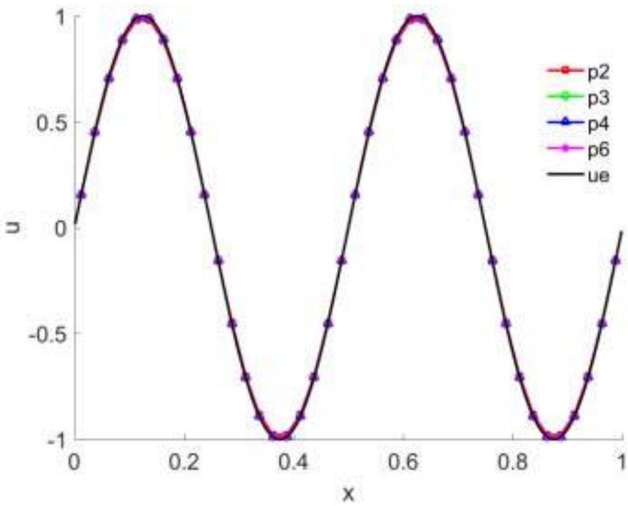
Solving by MATLAB, we can obtain the numerical format and data comparison of each order. It can be shown by Figure 3 and Table 1. The codes of solving this example will be in appendix.



(a) The calculation result of $\Delta x = 0.1$



(b) The calculation result of $\Delta x = 0.05$



(c) The calculation result of Δx = 0.01

Figure 3. Numerical calculation results compare with exact solution of example 1

Table 1. Error and order of example 1 with different Δx

format	Δx	Δt	N_x	L_1 Error	order	L_∞ Error	order
Second-order	1/20	Δx/10	20	3.61E-02		5.58E-02	
	1/40		40	6.00E-03	2.59	9.40E-03	2.57
	1/80		80	1.08E-03	2.47	1.70E-03	2.47
	1/160		160	2.17E-04	2.32	3.42E-04	2.31
	1/320		320	4.86E-05	2.16	7.48E-05	2.19
	1/640		640	1.11E-05	2.13	1.74E-05	2.10
third-order	1/20	Δx/20	20	6.69E-04		1.03E-03	
	1/40		40	8.17E-05	3.03	1.29E-04	3.00
	1/80		80	1.03E-05	2.99	1.61E-05	3.00
	1/160		160	1.28E-06	3.01	2.02E-06	2.99
	1/320		320	1.61E-07	2.99	2.52E-07	3.00
	1/640		640	2.01E-08	3.00	3.15E-08	3.00
fourth-order	1/20	Δx/200	20	2.28E-05		3.52E-05	
	1/40		40	1.39E-06	4.04	2.16E-06	4.03
	1/80		80	8.68E-08	4.00	1.36E-07	3.99
	1/160		160	5.43E-09	4.00	8.52E-09	4.00
	1/320		320	3.43E-10	3.98	5.38E-10	3.99
	1/640		640	2.62E-11	4.04	4.10E-11	4.03
sixth-order	1/20	Δx/1000	20	1.85E-08		2.85E-08	
	1/40		40	2.97E-10	5.96	4.60E-10	5.95
	1/80		80	8.79E-12	5.08	1.36E-11	5.08
	1/160		160	6.15E-12	0.52	9.59E-12	0.50
	1/320		320	1.81E-11		1.88E-11	
	1/640		640				

Example 2: convection diffusion equation

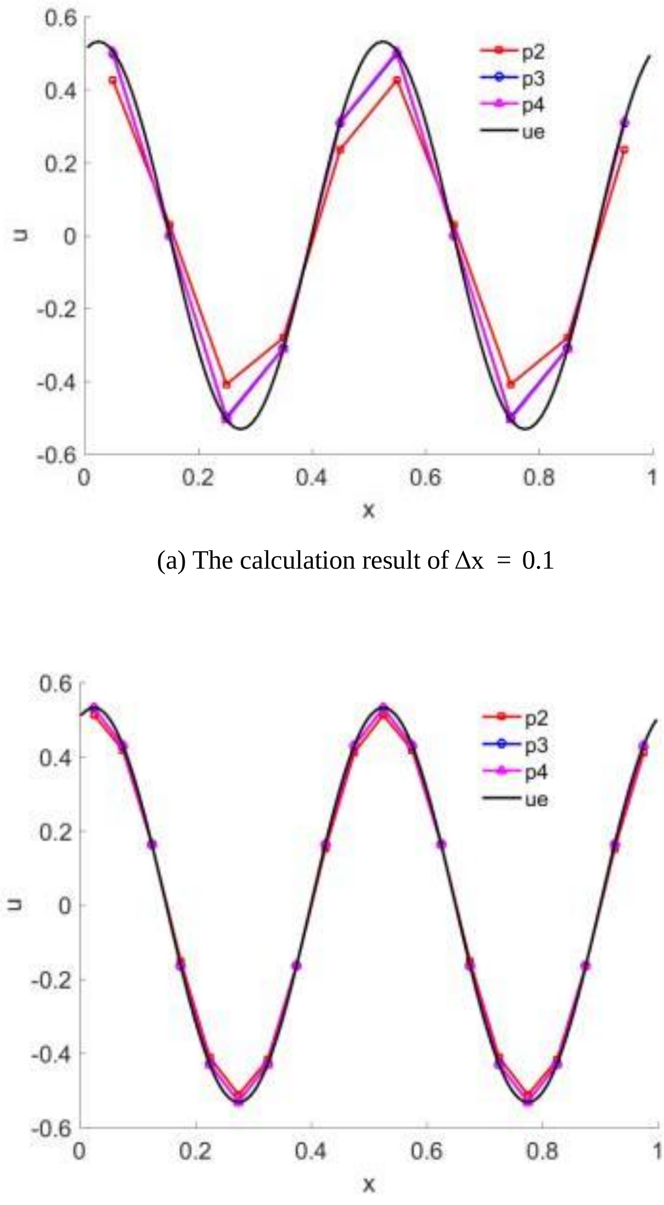
Consider a convection diffusion equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = v\dfrac{\partial^2 u}{\partial t^2} \\ u(x,0) = \sin(2\pi x) \\ u(0,t) = \exp(-4v\pi^2 t)\sin(-2\pi a t) \\ u(1,t) = \exp(-4v\pi^2 t)\sin\big(-2\pi(1-at)\big) \end{cases}$$

The exact solution to this problem is $u(x,t) = \exp(-4v\pi^2 t)\sin(-2\pi(x-at))$

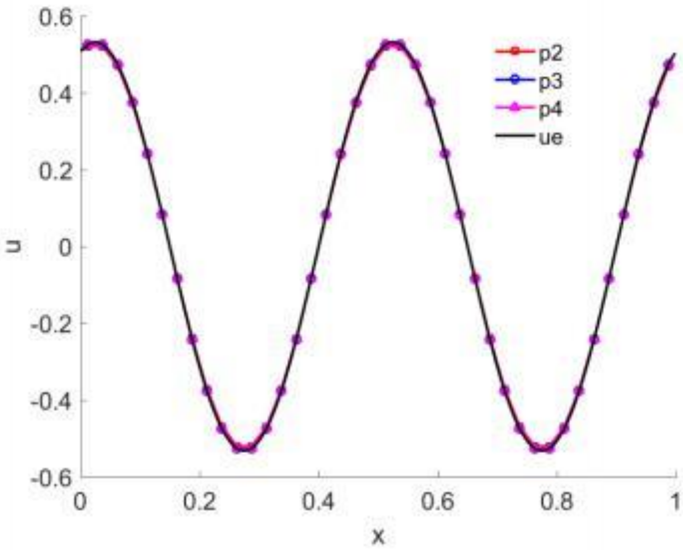
a=1.0, $\nu = 0.01$, time integration TVD RK3, $t_f = 0.4$.

Solving by MATLAB, we can obtain the numerical format and data comparison of each order. It can be shown by Figure 2 and Table 2. The codes of solving this example will be in appendix.



(a) The calculation result of Δx = 0.1

(b) The calculation result of Δx = 0.05



(c) The calculation result of $\Delta x = 0.01$

Figure 4. Numerical calculation results compare with exact solution of example 2

Table 2. Error and order of example 2 with different Δx

format	Δx	$\Delta t/(\Delta x)^2$	N_x	L_1 Error	order	L_∞ Error	order
Second-order	1/20	1	20	1.26E-02		1.95E-02	
	1/40	1	40	3.07E-03	2.04	4.85E-03	2.01
	1/80	1/2	80	7.72E-04	1.99	1.21E-03	2.00
	1/160	1/4	160	1.95E-04	1.99	3.06E-04	1.98
	1/320	1/8	320	4.89E-05	2.00	7.68E-05	1.99
	1/640		640				
third-order	1/20	1/2	20	7.23E-04		1.12E-03	
	1/40	1/2	40	9.05E-05	3.00	1.43E-04	2.97
	1/80	1/4	80	1.11E-05	3.03	1.75E-05	3.03
	1/160	1/8	160	1.38E-06	3.01	2.17E-06	3.01
	1/320	1/16	320	1.72E-07	3.00	2.70E-07	3.01
	1/640		640				
fourth-order	1/20	1/2	20	3.61E-05		5.58E-05	
	1/40	1/4	40	2.05E-06	4.14	3.22E-06	4.12
	1/80	1/8	80	1.18E-07	4.12	1.86E-07	4.11
	1/160	1/16	160	7.07E-09	4.06	1.11E-08	4.07
	1/320	1/32	320	4.55E-10	3.96	7.15E-10	3.96
	1/640		640				
sixth-order	1/20		20				
	1/40		40				
	1/80		80				
	1/160		160				
	1/320		320				
	1/640		640				

3. Application

In this chapter, we will induce the work and application of limiters, and research the influence of limiters of formats in different situations.

3.1 Limiters' Works and Different Limiters

The work of the limiter is generally divided into two parts: catching the troubled cell and reconstructing the troubled cell.

Then we introduce some common limiters.

1) Limiter TVB based on minmod function (TVB)

We provide an overview of minmod limiter in the one -dimensional scalar case. First, denote that $\bar{u}_j = \frac{1}{\Delta x_j}\int_{I_j} u dx$ to be the cell average of the solution u.

Further denote $\tilde{u}_j = u^-_{j+\frac{1}{2}} - \bar{u}_j$, $\tilde{\tilde{u}}_j = \bar{u}_j - u^+_{j+\frac{1}{2}}$.

$\tilde{u}_j$ and $\tilde{\tilde{u}}_j$ are modified either by the usual minmod limiters,

$\tilde{u}_j^{(mod)} = m(\tilde{u}_j, \Delta_+\bar{u}_j, \Delta_-\bar{u}_j)$, $\tilde{\tilde{u}}_j^{(mod)} = m(\tilde{\tilde{u}}_j, \Delta_+\bar{u}_j, \Delta_-\bar{u}_j)$,

where $\Delta_+\bar{u}_j = \bar{u}_{j+1} - \bar{u}_j, \quad \Delta_-\bar{u}_j = \bar{u}_j - \bar{u}_{j-1}$,
with the minmod function m defined by

$$m(a_1, \dots, a_1) = \begin{cases} s \min_{1\le j\le l} |a_j| & if\ s = sign(a_1) = \cdots = sign(a_1), \\ 0, & otherwise, \end{cases}$$

or by the TVB modified minmod function

$$\tilde{m}(a_1, \dots, a_1) = \begin{cases} a_1 & if\ |a_1| \le Mh^2, \\ m(a_1, \dots, a_1), & otherwise, \end{cases}$$

The TVB parameter M should be chosen depending on the solution of the problem.

2) Limiter WENO

To make the weighted intrinsically non-oscillating scheme easier to understand, we divide it into five steps.
(For a troubled cell $K = I_I \times I_I$)

Step 1: Note the polynomial on K to be p_0, the right, left, up and down polynomials of its four neighbors are denoted as p_1, p_2, p_3, p_4.

Step 2: Set $\bar{p}_0 = \frac{1}{|K|}\int_K p_0 dx$ to be the average value of the cell. We update the constant terms of p_1, p_2, p_3, p_4 such that their new cell averages are the same as p_0, that is: $\tilde{p}_i = p_i - \bar{p}_i + \bar{p}_0, i = 1,2,3,4$. For convenience, we take $\tilde{p}_0 = p_0$. The point of this is that we want to express p_0^{new} as a convex combination of $\tilde{p}_0, \tilde{p}_1, \dots, \tilde{p}_4$, and since their cell average is all $\bar{p}_0$, $\bar{p}_0^{new} = \bar{p}_0$.

This reconstruction does not change the cell average of the polynomial over K.

Step 3: Computed smooth indicator

$$\beta_j = \sum_{|l|=1}^{k} |K|^{|l|-1} \int_K \left(\frac{\partial^l p_i(x,y)}{\partial x^{l_1}\partial y^{l_2}}\right)^2 dxdy, j = 0, 1,2,3,4$$

Where $l = (l_1, l_2)$ is multi-index.

Step 4: Calculated weight

$\omega_j = \frac{\gamma_j}{(\beta_j+\varepsilon)^2}$, $j = 0,1,2,3,4$. ε is a very small number, take 10^{-6}, γ_i is linear weight, we can choose

$\gamma_0 = 0.996$, $\gamma_1 = \cdots = \gamma_4 = 0.001$.By normalization, we get:

$$\tilde{\omega}_j = \frac{\omega_j}{\sum_{i=0}^{4} \omega_i}$$

Step 5: Obtain the reconstructed polynomial over K:

$$p_0^{new} = \tilde{\omega}_0\tilde{p}_0 + \tilde{\omega}_1\tilde{p}_1 + \tilde{\omega}_2\tilde{p}_2 + \tilde{\omega}_3\tilde{p}_3 + \tilde{\omega}_4\tilde{p}_4$$

Note: About the linear weight γ_i, if take a large γ_0, it can maintain the accuracy better, since when $\gamma_0 = 1$, a limiter is unnecessary. But if take a smaller γ_0, it can better limit shock.

3) Limiter WBAP (paper (Li W, Ren YX, Lei G & Luo H, 2011; Li W & Ren YX, 2012))

3.2 Some Examples

Example 1: Burgers equation in 2D

The two-dimensional burgers equation is of the form:

$u_t + \left(\frac{u^2}{2}\right)_x + \left(\frac{u^2}{2}\right)_y = 0, u(x, y, 0) = u^0(x, y)$, or $u_t + uu_x + uu_y = 0$.

Create a transformation substitute $\xi = x + y, \eta = x - y$, then

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = u_\xi + u_\eta$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = u_\xi - u_\eta$$

That is, $u_x + u_y = 2u_\xi$, it implies that $u_t + 2uu_\xi = 0$.

That is to say, u for $\xi = x + y$ satisfies a relationship similar to a 1D Burgers equation. Assume that $u(\xi, 0) = u^0(\xi)$, then for time t, on the line $x + y = \xi$, the value of u is $u^0(\xi^*)$, where $\xi^* + 2t \cdot u^0(\xi^*) = \xi$.

Consider a simple example: take the computing domain as $[0,4]^2$, boundary condition is periodic, initial value be $u^0(x, y) = \sin\frac{\pi(x+y)}{2}$.

When $T = \frac{-1}{2 \min u_0{}'(\xi)} = \frac{1}{\pi}$, there are two shock wave structures on $\xi = 2$ and $\xi = 6$.

By using Newton iteration method, we can get approximate true solution.

Let $t_{end} = 1.5/\pi$.

4. Conclusion

In conclusion, a novel compact high-order finite volume method, is developed based on the principles of the compact difference method. The validity of the proposed scheme is rigorously analyzed, and its effectiveness is demonstrated through comprehensive numerical experiments. These experiments yield favorable results, substantiating both the theoretical analysis and the practical applicability of the scheme.

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