Paradigm Academic Press
Innovation in Science and Technology
ISSN 2788-7030
OCT. 2022 VOL.1 NO.3

A Multistep Method for a Special Class of Second—Order Differential Equations

S. O. Ayinde[1], M. O. Oke[1], R. B. Ogunrinde[1], A. A. Obayomi[1],
S. N. Ogunyebi[1], S. E. Fadugba[1] & O.E. Abolarin[2]

[1] Department of Mathematics, Ekiti State University, Ado Ekiti, Ekiti State, Nigeria

[2] Mathematics Department, Federal University Oye Ekiti, Ekiti State, Nigeria

Correspondence: S. O. Ayinde, Department of Mathematics, Ekiti State University, Ado Ekiti, Ekiti State, Nigeria.

doi: 10.56397/IST.2022.10.02

Abstract

A multi–step numerical method for the solution of second order ordinary differential equation was developed by interpolating in a finite range with a basis function. The basis function consists of a combination of exponential and trigonometric functions to ensure that such problems possess unique and continuously differentiable solutions. The method has been tested and found to be reliable, efficient and less tedious than other multi-step methods which require reduction of higher order equations into several first order equations. The method was applied to some special second order equations arising from mechanics and engineering problems. The requisite numerical properties were obtained.

Keywords: special higher-order equations, numerical method, basis function, linear multistep method

1. Introduction

A second order differential equation can be reduced into a system of first order ODEs. Such systems have been solved using several analytic and numerical approaches in the past. Popular among such numerical methods are the predictor–corrector and other finite difference methods. These methods have been used by many scholars like Awoyemi and Kayode (2005), Kayode and Adeyeye (2011) and others. Recently, Block Methods become more widely accepted as can be found in the works of Jator (2007), Omar and Raft (2016) and Abolarin et al. (2020).

A second order ordinary differential equation is of the form

$$y''(x) = f(x, y(x), y'(x)) \tag{1}$$

which may be reduced to the first order system by letting

$$u' = r, r' = f(x, u, r)$$

where

$$u = y, r = y'.$$

However, the equation (1) can be stated in the special form

$$y''(x) = f(x, y(x)) \tag{2}$$

from the special higher order equations of the form $y^{(m)}(x) = f(x, y(x))$. (Frank A., & Elliot M., 2009)

The standard special initial value problem for (2) is

$$y''(x) = f(x, y),\ \ y(a) = \varphi \ and$$
$$y'(a) = \omega \tag{3}$$

The general solution of a first-order differential equation contains one arbitrary constant, and a single additional condition called an initial condition enough to fix the value of the constant and hence determine the particular solution. However, the second-order differential equation contains two arbitrary constants, and two additional conditions as required (see (The Open University., 2013)).

The general computational algorithm for determining the sequence $\{y_n\}$ for numerical solution at the n^th point takes the form of a linear relationship between y_{n+j}, f_{n+j}, $j = 0, 1, \dots .k$, which is referred to as linear multistep method of step-number k, or a linear k-step method. The general linear multistep method can then be written as class of difference equations

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (4)$$

Where h is a step size parameter, $f_{0.}\, f_1, f_2, \dots$ are the values of a given function $f(x, y)$ at equidistant arguments, α_j and β_j are constants. We will assume that $\alpha_k \neq 0$, both α_0 and β_0 are not zeros. (Lambert, J.D., 1972).

For the purpose of this research, the special higher order of the form (2) shall be considered with the initial conditions given by (3). Such second order ordinary differential equation arises in a number of important applications, particularly in Mechanics and Engineering.

2. Derivation of the Numerical Method

Let us assume that the theoretical solution $y(x)$ to (3) can be represented in the interval $[x_{n,}\, x_{n+1}\ ], n \geq 0$ by the basis function,

$$F(x) = \alpha_1 \beta e^{\beta x} + \alpha_2 B^x \log B - \alpha_3 sinx \qquad (5)$$

Where $\alpha_1, \alpha_2, and\ \alpha_3$ are real undetermined coefficients, $B\ and\ \beta$ are constants. The interval under consideration is $x \in [0,1]$. The basis function (5) is the second order of the basis function which has been used as a basis function to develop numerical scheme by (Oguunrinde R. B. & Ayinde S O., 2017) to solve initial value problems in first order ordinary differential equations.

Let y_n be the numerical estimate to the theoretical solution $y(x)$ at the nth point, and can be represented by the function $f_n = f(x_n, y_n)$. Define the mesh points as follows: $x_n = a + nh, x_{n+1} = a + (n + 1)h, n = 0, 1, 2, \dots$

Impose some constraints on the interpolating function (5) in order to get the undetermined coefficients.

Let the interpolating function coincide with the theoretical solution at $x = x_n\ and\ x = x_{n+1}$. This required that

$$F(x_n) = \alpha_1 \beta e^{\beta x_n} + \alpha_2 B^{x_n} \log B - \alpha_3 sinx_n \qquad (6)$$

and

$$F(x_{n+1}) = \alpha_1 \beta e^{\beta x_{n+1}} + \alpha_2 B^{x_{n+1}} \log B - \alpha_3 sinx_{n+1} \qquad (7)$$

and the derivatives of the basis function coincide with the differential equation as well as its first, second, and third derivatives with respect to $x\ \ at\ x = x_n$.

Denote the *i*-th total derivatives of $f(x, y)$ with respect to x with $f^{\ i}$ such that

$$F'(x_n) = f_n,\ F''(x_n) = f_n^1,$$
$$F'''(x_n) = f_n^2 \qquad (8)$$

This implies that,

$$f_n = \alpha_1 \beta^2 e^{\beta x_n} + \alpha_2 B^{x_n} (\log B)^2 - \alpha_3 cosx_n \qquad (9)$$

$$f_n{}^1 = \alpha_1 \beta^3 e^{\beta x_n} + \alpha_2 B^{x_n} (\log B)^3 + \alpha_3 sinx_n \qquad (10)$$

$$f_n{}^2 = \alpha_1 \beta^4 e^{\beta x_n} + \alpha_2 B^{x_n} (\log B)^4 + \alpha_3 cosx_n \qquad (11)$$

Solving for $\alpha_1,\ \alpha_2, and\ \alpha_3$ from the system of equation (9) to (11), we have

$$\begin{pmatrix} \beta^2 e^{\beta x_n} & B^{x_n}(logB)^2 & -cosx_n \\ \beta^3 e^{\beta x_n} & B^{x_n}(logB)^3 & sinx_n \\ \beta^4 e^{\beta x_n} & B^{x_n}(logB)^4 & cosx_n \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} f_n \\ f_n{}^1 \\ f_n{}^2 \end{pmatrix} \qquad (12)$$

Taking this as system of equations, $AX = B$, it gives

$$\alpha_1 = \frac{\begin{array}{c}[f_n(\log B)^3 - f_n^1(logB)^2 - f_n^1(logB)^4 + f_n^2(logB)^3]cosx_n \\ +[f_n^2(logB)^2 - f_n(\log B)^4]sinx_n\end{array}}{\begin{array}{c}e^{\beta x_n}[(\beta^2(logB)^3 - \beta^3(logB)^2 - \beta^3(logB)^4 + \beta^4(logB)^3)cosx_n \\ +(\beta^4(logB)^2 - \beta^2(logB)^4)sinx_n]\end{array}} \qquad (13)$$

$$\alpha_2 = \frac{\beta^2\left(f_n^1 cosx_n - f_n^2 sinx_n\right) - \beta^3 cosx_n(f_{n+}f_n^2) + \beta^4(f_n sinx_n + f_n^1 cosx_n)}{\begin{array}{c}B^{x_n}[(\beta^2(logB)^3 - \beta^3(logB)^2 - \beta^3(logB)^4 + \beta^4(logB)^3)cosx_n \\ +(\beta^4(logB)^2 - \beta^2(logB)^4)sinx_n]\end{array}} \qquad (14)$$

$$\alpha_3 = \frac{f_n\left(\beta^3(logB)^4 - \beta^4(logB)^3\right) - f_n^1\left(\beta^4(logB)^2 - \beta^2(logB)^4\right) + f_n^2\left(\beta^2(logB)^3 - \beta^3(logB)^2\right)}{[(\beta^2(logB)^3 - \beta^3(logB)^2 - \beta^3(logB)^4 + \beta^4(logB)^3)cosx_n + (\beta^4(logB)^2 - \beta^2(logB)^4)sinx_n]} \tag{15}$$

Since $F(x_{n+1}) = y(x_{n+1})$ and $F(x_n) = y(x_n)$

Implies that $y(x_{n+1}) = y_{n+1}$ and $y(x_n) = y_n$, therefore,

$$F(x_{n+1}) - F(x_n) = y_{n+1} - y_n \tag{16}$$

Therefore,

$$y_{n+1} - y_n = \alpha_1\beta\left[e^{\beta x_{n+1}} - e^{\beta x_n}\right] + \alpha_2 logB[B^{x_{n+1}} - B^{x_n}] + \alpha_3[sinx_n - sinx_{n+1}] \tag{17}$$

Recall that $x_n = a + nh, \quad x_{n+1} = a + (n+1)h$ with $n = 0,1,2$ ... by expansion of (17)

$$y_{n+1} - y_n = \alpha_1\beta e^{\beta x_n}\left(e^{\beta h} - 1\right) + \alpha_2 B^{x_n}(B^h - 1)logB + \alpha_3(sinx_n - \sin(x_n + h)) \tag{18}$$

Let

$$M = \alpha_1\beta e^{\beta x_n}\left(e^{\beta h} - 1\right)$$

$$L = \alpha_2 B^{x_n}(B^h - 1)logB$$

$$N = \alpha_3(sinx_n - \sin(x_n + h))$$

Substituting for α_1, α_2, and α_3 from (13, 14, 15) into (18), gives,

$$y_{n+1} = y_n + M + L + N \tag{19}$$

where

$$M = \beta\left(e^{\beta h} - 1\right)\frac{[f_n(log\,B)^3 - f_n^1(logB)^2 - f_n^1(logB)^4 + f_n^2(logB)^3]cosx_n + [f_n^2(logB)^2 - f_n(log\,B)^4]sinx_n}{[(\beta^2(logB)^3 - \beta^3(logB)^2 - \beta^3(logB)^4 + \beta^4(logB)^3)cosx_n + (\beta^4(logB)^2 - \beta^2(logB)^4)sinx_n]}$$

$$L = (B^h - 1)logB\frac{\beta^2(f_n^1 cosx_n - f_n^2 sinx_n) - \beta^3 cosx_n(f_n + f_n^2) + \beta^4(f_n sinx_n + f_n^1 cosx_n)]}{B^{x_n}[(\beta^2(logB)^3 - \beta^3(logB)^2 - \beta^3(logB)^4 + \beta^4(logB)^3)cosx_n + (\beta^4(logB)^2 - \beta^2(logB)^4)sinx_n]}$$

$N = (sinx_n - \sin(x_n + h))$ X

$$\frac{f_n\left(\beta^3(logB)^4 - \beta^4(logB)^3\right) - f_n^1\left(\beta^4(logB)^2 - \beta^2(logB)^4\right) + f_n^2\left(\beta^2(logB)^3 - \beta^3(logB)^2\right)}{[(\beta^2(logB)^3 - \beta^3(logB)^2 - \beta^3(logB)^4 + \beta^4(logB)^3)cosx_n + (\beta^4(logB)^2 - \beta^2(logB)^4)sinx_n]} \tag{19}$$

Equation (19) is the numerical method derived for the solution of second - order differential equations.

3. Analysis of the Properties of the Numerical Method

Consider the derived numerical method (19) in form of multi-step method, we shall establish the numerical algorithm which can be expressed in the form

$$y_{n+1} = y_n + h\emptyset(x_n, y_n; h) \tag{20}$$

Where $\emptyset(x_n, y_n; h)$ is the increment function.

Expanding $e^{\beta h}$ and B^h in Taylor's series (to the fifth term), also the Taylor's series of hyperbolic function of $cosh$ (to the fourth terms), i.e.

$$e^{\beta h} = \sum_{r=0}^{\infty}\frac{(\beta h)^r}{r!} \tag{21}$$

$$B^h = \sum_{r=0}^{\infty} h^r(logB)^r \tag{22}$$

Also $cos\,h$ and $sin\,h$ gives,

$$cos\,h = \sum_{r=0}^{\infty}\frac{h^{2r}}{(2r)!} \tag{23}$$

$$Sin\,h = \sum_{r=0}^{\infty}\frac{h^{(2r+1)}}{(2r+1)!} \tag{24}$$

substitute (21, 22, 23,24) into (19) and expand,

Let $f_n = fn$, $f_n^1 = fn1$ and $f_n^2 = fn2$, therefore, we have

$$y_{n+1} = y_n + h\{Af_n + Bf_n^1 + Cf_n^2\} \tag{25}$$

Where

$A = \Big[\frac{h}{2}\beta^3 logB^3 + \frac{h^2}{3}\beta^4 logB^3 + \frac{h^3}{24}\beta^5 logB^3 - h\beta^3 logB^3 - h^2\beta^3 logB^4 - h\beta^3 logB^5 - \beta^3 logB^4 - \frac{h^2}{6}\beta^3 logB^4 - \frac{h^4}{120}\beta^3 logB^4 - \frac{h^6}{5040}\beta^3 logB^4 + \beta^4 logB^3 + \frac{h^4}{120}\beta^4 logB^3 + \frac{h^6}{5040}\beta^4 logB^3\Big] cosx_n + \Big[-\beta^2 logB^4 - \frac{h^2}{6}\beta^4 logB^4 - \frac{h^3}{24}\beta^5 logB^4 + \beta^4 logB^2 + h\beta^4 logB^3 + h^2\beta^4 logB^4 + h\beta^4 logB^5 + \frac{h^3}{24}\beta^3 logB^4 + \frac{h^5}{720}\beta^3 logB^4 - \frac{h}{2}\beta^4 logB^3 - \frac{h^3}{24}\beta^4 logB^3 - \frac{h^5}{720}\beta^4 logB^3\Big] sinx_n$

$B = \Big[-\frac{h}{2}\beta^3 logB^2 - \frac{h}{2}\beta^3 logB^4 - \frac{h^2}{6}\beta^4 logB^2 - \frac{h^2}{6}\beta^4 logB^4 - \frac{h^3}{24}\beta^5 logB^2 - \frac{h^3}{24}\beta^5 logB^4 + h\beta^2 logB^3 + h\beta^4 logB^3 + h\beta^2 logB^5 + h\beta^4 logB^5 + h^2\beta^2 logB^4 + h^2\beta^4 logB^4 + \frac{h^2}{6}\beta^2 logB^4 + \frac{h^4}{120}\beta^2 logB^4 + \frac{h^6}{5040}\beta^2 logB^4 + \frac{h^4}{120}\beta^4 logB^2 + \frac{h^6}{5040}\beta^4 logB^2\Big] cosx_n + \Big[-\frac{h}{2}\beta^2 logB^4 - \frac{h}{2}\beta^4 logB^2 - \frac{h^3}{24}\beta^2 logB^4 - \frac{h^3}{24}\beta^4 logB^2 - \frac{h^5}{720}\beta^2 logB^4 - \frac{h^5}{720}\beta^4 logB^2\Big] sinx_n$

$$C = \Big[-h\beta^3 logB^3 - h\beta^3 logB^5 - h^2\beta^3 logB^4 + \frac{h}{2}\beta^3 logB^3 + \frac{h^2}{6}\beta^4 logB^3 + \frac{h^2}{6}\beta^2 logB^3 + \frac{h^2}{6}\beta^3 logB^2 + \frac{h^3}{24}\beta^5 logB^3 - \frac{h^4}{120}\beta^4 logB^3 + \frac{h^4}{120}\beta^3 logB^2 - \frac{h^6}{5040}\beta^2 logB^3 + \frac{h^6}{5040}\beta^3 logB^2\Big] cosx_n + \Big[-h\beta^2 logB^3 - h\beta^2 logB^5 - h^2\beta^4 logB^4 + \frac{h}{2}\beta^2 logB^3 + \frac{h^2}{6}\beta^4 logB^2 + \frac{h^3}{24}\beta^5 logB^2 + \frac{h^3}{24}\beta^2 logB^3 - \frac{h^3}{24}\beta^3 logB^2 + \frac{h^5}{720}\beta^2 logB^3 - \frac{h^5}{720}\beta^3 logB^2\Big] sinx_n$$

Therefore, (25) is the increment function of (19).

3.1 Consistency of the Numerical Method

According to (Dahlquist G., 1956), second order ordinary differential equations of the form (3) which is an m – step linear Multistep Method that needs $m - 1$ additional starting values. Consider a class of difference equations (4) of the form

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i}$$

where h is a parameter and $f_0, f_1, f_2, \ldots$ are values of a given function $f(x, y)$ at equidistant arguments $x_n = a + nh$. i.e $f_n(x_n, y_n)$. The initial value problem for an ordinary differential equations $y' = f(x, y), y(a) = y_0$ has some particular formulas of the form

$$y_{n+1} - y_n = hf_n \quad (26)$$

However, as y_n tends to the solution $y(x)$ when $h \to 0$ and $n \to \infty$, so $x_n = a + nh$ provided that $f(x, y)$ is continuous and satisfies a Lipschitz condition. Two formulas that can be utilized are

$$y_{n+1} - y_n = \frac{1}{2}h(f_{n+1} + f_n) \quad (27)$$

$$y_{n+2} - y_n = \frac{1}{3}h(f_{n+2} + 4f_{n+1} + f_n \quad (28)$$

Equation (27) is based on quadratic formula known as Trapezoidal rule and (28) is based on Simpson's rule. In order that the difference equation (4) should be useful for Numerical method, it is necessary that it satisfied with good accuracy by the solution of $y''(x) = f(x, y),\ y(a) = \varphi\ and\ y'(a) = \omega$, when h is small, for an arbitrary function $f(x, y)$. It follows from this that the value of the expression $L[y(x)]$ defined by

$$L[y(x)] = \sum_{i=0}^{k}[\alpha_i y(x + ih) - h\,\beta_i y'(x + ih)] \quad (29)$$

should be small when h is small, for all sufficiently regular function $y(x)$. This imposes restrictions on the coefficients α_i, β_i in (4). By expanding the terms in (29) into powers of h we see that

$$L[y(x)] = \sum_{q=0}^{r+1} C_q h^q y^q + O(h^{r+2}) \quad (30)$$

Where

$$C_q = \begin{cases} \sum_{i=0}^{k} \alpha_i & if q = 0 \\ \sum_{i=0}^{k} \frac{(-i)^q}{q!}\alpha_i + \frac{(-i)^{q-1}}{(q-1)!}\beta_i\ if\ q > 0 \end{cases} \quad (31)$$

for arbitrary $y(x)$ if and only if the following $p + 1$ linear which have order p and can be written as

$$\rho(e^h) - h\sigma(e^h) = O(h^{p+1}) \quad as\ h \to 0 \quad (32)$$

Since a numerical method is consistent if it has order at least one, which is the case if $\rho(1) = 0$ and $\rho'^{(1)} = \sigma(1)$. The numerical method (25) is of order $p, p > 0$. Hence it is consistent.

3.2 Zero Stability of the Numerical Method

Definition 1 (Jain, M.K., Iyengar, S.R.K. & Jain, R.K., 2012)

The Multistep Method (4) is said to be zero stable if all solutions of the homogeneous linear difference equation

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = 0 \tag{33}$$

are bounded for all n.

Consider the linear difference equation

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \emptyset_n \ , \ n = n_0 \tag{34}$$

where $\alpha_0, \alpha_1, \dots, \alpha_k$ are constants independent of $n, \alpha_0 \neq 0, \alpha_k \neq 0$ and $\{\emptyset_n\}, n = n_0$ is a known sequence. A solution of this difference equation is a sequence of values $\{y_n\}, \ n = n_0$ that satisfies (34) identically for all $, n = n_0$. The general solution to (4) in the from (19) can be written as

$$y_n = \hat{y}_n + \varphi_n \tag{35}$$

where $\hat{y}_n$ is the general solution of the associated homogeneous equation (33) and φ_n is a particular solution of (34). To determine $\hat{y}_n$, use

$$\hat{y}_n = r^n \tag{36}$$

substituting (36) into (33) yields

$$\sum_{i=0}^{k} \alpha_i r^{n-i} = 0 \text{ , dividing by } r^{n-k} \quad \text{yields}$$

$$\rho(r) = \sum_{i=0}^{k} \alpha_i r^{n-i} = 0 \tag{37}$$

hence, the k roots of $\rho(r)$ is $r_1, r_2, \dots, r_k$.

In general, the multistep method (4) is zero stable for the initial value problem of second order (1) for sufficiently small h if there exists some constant m independent of h such that

$$|y_n - \hat{y}_n| \leq M \max_{0 \leq i \leq m-1} |y_n - \hat{y}_n|$$

for all n with $x_0 \leq x_k \leq x_n$. More plainly, a method is zero stable for a particular problem if errors in the starting values are not magnified in an unbounded fashion, this is shown in the test equation below.

Theorem 1 (Lambert, J.D., 1972)

A linear multistep method is zero stable for any initial value problem of second order provided it satisfies the root condition

(i) all roots $\rho(r) = 0$ lie in the unit disk. i.e $|r| \leq 1$

(ii) any roots in the unit circle ($|r| \leq 1$) are simple.

3.3 Convergence of the Numerical Method

Definition 2 (Lambert, J.D., 1972)

The linear multistep method $\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$ where α_j and β_j are constants, assume that $\alpha_k \neq 0$ and that not both α_0 and β_0 are zero, is said to be Convergent if for all initial value problem $y(a) = \varphi \ and \ y'(a) = \omega$, subject to the hypothesis of Lipschitz condition, we have $\lim_{h \to 0} y_n = y(x_n)$ holds for all $x \in [a, b]$, and all solutions $\{y_n\}$ of the difference equation $y''(x) = f(x, y(x))$, satisfying the conditions $y_\tau = \varphi_\tau(h)$ for which $\lim_{h \to 0} \varphi_\tau(h) = \varphi, \tau = 0,1,2, \dots k - 1$

Theorem 2 (Dahlquist Equivalence Theorem)

Suppose an m – step multistep method (19) applied to an initial value problem on $[x_0, x_n]$ with constant starting values $y_n - y(x_n)$ for $x_n = x_0 + nh$, $n = 0, 1, \dots, m - 1$ as $h \to 0$. This method is convergent i.e $y(x_n) \to y(x)$ for all $x \in [x_0, x_n]$ as $h \to 0$ if and only if the method is consistent and zero stable. Since the exact solution is sufficiently smooth, $y(x) \in C^{p+1}(x_0, x_n)$ and the multistep method is order p, then

$$y(x_k) - y_k = O(h^p) \tag{38}$$

for all $x_k \in [x_0, x_n]$ as it satisfy the following theorem.

Theorem 3 (Fatunla, S.O., 1988)

Let the increment function $y_{n+1} = y_n + h\{Af_n + Bf_n^1 + Cf_n^2\}$ as drawn in (32) be continuous in each of its arguments for $(x, y) \in R^{m+1}$ and $0 < h \leq h_0$, and, in addition, let $\emptyset$ satisfy a Lipschitz condition of order one with respect to y. Then $y_{n+1} = y_n + h\emptyset(x_n, y_n; h)$ is convergent if and only if it is consistent and stable.

3.4 Stability Analysis of the Numerical Method

Definition 3 (Fatunla, S.O., 1988)

The Linear Multistep Method is said to be absolutely stable for a given z if $|\lambda_r| \leq 1, \quad r = 1(1)k,$ and the region of absolute stability (RAS) is the set

$$l = \{z \in C: |\lambda_r| \leq 1, r = 1(1)k$$

Consider the increment function (25) to the numerical method (19) of the form

$$y_{n+1} = y_n + h\{Af_n + Bf_n^1 + Cf_n^2\}$$

Subjected to expansion in h, grouping and collecting the like terms gives

$$y_{n+1} = y_n + hf_n + \frac{1}{2}h^2 f_n^1 + \frac{1}{6}h^3 f_n^2 \tag{39}$$

According to (Awoyemi, D. O., & Kayode, S. J., 2005), the stability characteristics of the Multistep Method equation

$$\rho(E)y_n = h^2\sigma(E)f_n \tag{40}$$

is normally investigated by its application to the scalar test equation

$$y_n'' = \lambda^2 y_n \tag{41}$$

whose resultant finite equation has the characteristic equation

$$\pi(z, R) = \rho(R) - z^2\rho(R), \; z = i\lambda h \tag{42}$$

$$= \sum_{i=0}^{n} Q_i(z^2)R^{k-i}$$

Where $Q_i(z^2)$ are polynomials in z^2.

Hence, using the test equation,

$$y_n'' = \lambda^2 y_n = f_n, y_n''' = \lambda^3 y_n = f_n^1, y_n^{iv} = \lambda^4 y_n = f_n^2 \tag{43}$$

Substituting (41) into (39), and expand gives

$$y_{n+1} = y_n + \frac{1}{2}h^2\lambda^2 y_n + \frac{1}{6}h^3\lambda^3 y_n + \frac{1}{24}h^4\lambda^4 y_n \tag{44}$$

$$\frac{y_{n+1}}{y_n} = 1 + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4 \tag{45}$$

If $z = \lambda h$ and $\frac{y_{n+1}}{y_n} = \mu(z)$, (45) becomes

$$\mu(z) = 1 + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 \tag{46}$$

which is the stability function of the method (19)

Definition 4 (Fatunla, S.O., 1988)

The Linear Multiple Method equation (39) is said to be P-stable if its interval of periodicity is $(0, \infty)$.

Therefore the region of absolute stability of the method (19) is verified using (46) as $-4.0 < u < 0.5$ and $-4.0 < v < 4.0$ in the region $0 \leq \theta < 2 * pi$. The method is absolutely stable at the shaded points as seen in the graph below.

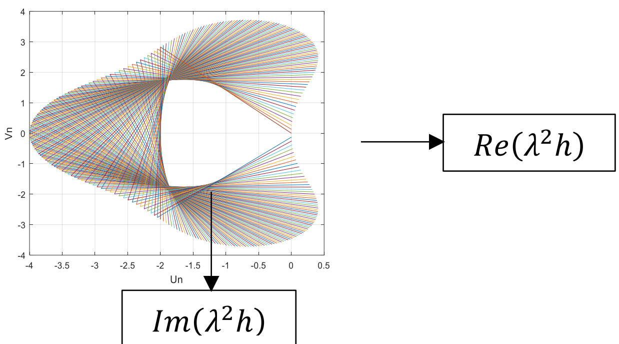

Figure 1. The shaded portion showing the Region of Absolute Stability for the Numerical Method (19)

4. The Implementation and Results of the Numerical Method

Problem 1: Using Numerical Method (19) to solve the initial value problem

$y'' + 9 = x + 2,\ y(0) = -1,\ y'(0) = 1$, which is non – autonomous , *in the interval* $0 \leq x \leq 1$, The analytical solution, $y(x) = \frac{1}{6}x^3 - \frac{7}{2}x^2 + x - 1$

Table 1. Results of problem 1, for $h = 0.025$

Xn	Numerical Solution	Exact Solution	Absolute Error
0.000	-1.0000000000	-1.0000000000	0.0000000000
0.025	-0.9771642416	-0.9771848958	0.0000206543
0.125	-0.9265794390	-0.9293619792	0.0027825401
0.250	-0.9649405317	-0.9661458333	0.0012053017
0.375	-1.1057111778	-1.1083984375	0.0026872597
0.500	-1.3494974679	-1.3541666667	0.0046691987
0.625	-1.6943573540	-1.7014973958	0.0071400418
0.750	-2.1382909469	-2.1484375000	0.0101465531
0.875	-2.6792086005	-2.6930338542	0.0138252536
1.000	-3.3148564212	-3.3333333333	0.0184769122

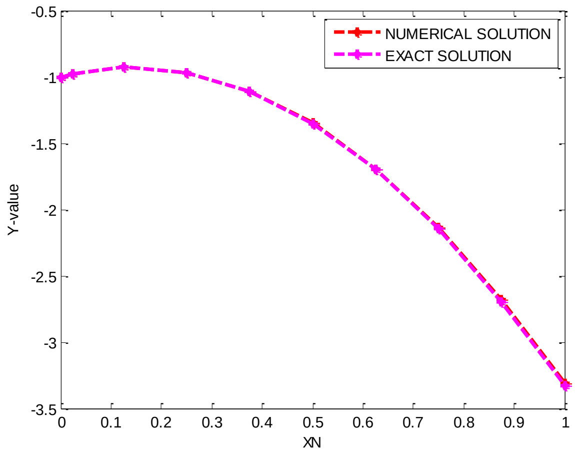

Figure 2. The graph of the solutions to problem 1 on table 1

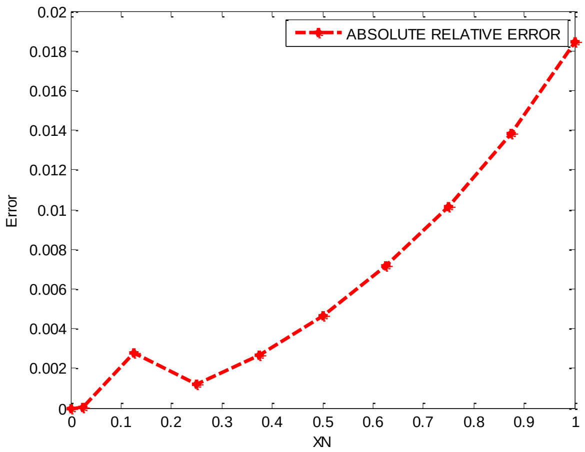

Figure 3. The graph of the absolute error to problem 1 on table 1

Problem 2: Using Numerical Method (19) to solve the initial value problem
$y'' + y = 0,\ y(0) = 3,\ y'(0) = 2,\ in\ the\ interval\ 0 \le x \le 1,$ which is autonomous. The analytical solution $y(x) = 2sinx + 3cosx.$

Table 2. Results of problem 2, for $h = 0.01$

Xn	Numerical Solution	Exact Solution	Absolute Error
0.00	3.0000000000	3.0000000000	0.0000000000
0.10	3.1936534241	3.1846793291	0.0089740950
0.20	3.3546994080	3.3375383951	0.0171610129
0.30	3.4816171838	3.4570498807	0.0245673031
0.40	3.5732493919	3.5420196666	0.0312297253
0.50	3.6288267538	3.5915987629	0.0372279909
0.60	3.6479997630	3.6052917915	0.0427079715
0.70	3.6308906775	3.5829619363	0.0479287412
0.80	3.5782041636	3.5248323098	0.0533718538
0.90	3.4915338817	3.4314837241	0.0600501576
1.00	3.3745722863	3.3038488872	0.0707233990

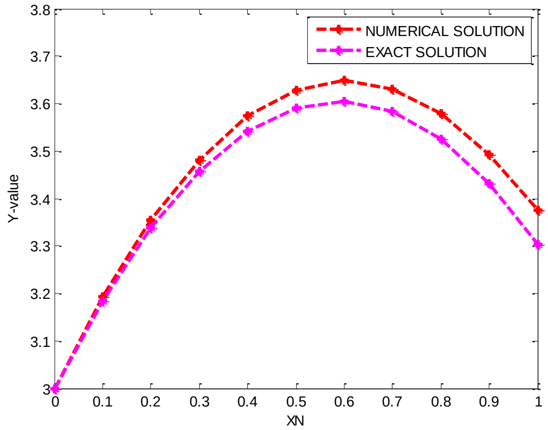

Figure 4. The graph of the solutions to problem 2 on table 2

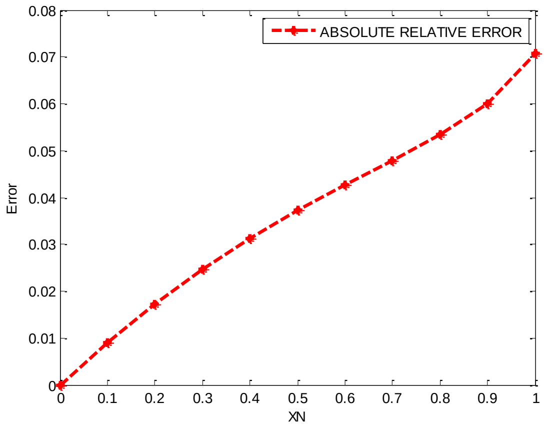

Figure 5. The graph of the absolute error to problem 2

Problem 3: Using Numerical Method (19) to solve the initial value problem
$y'' + y = sinx,\ y(0) = 1,\ y'(0) = 3,\ in\ the\ interval\ 0 \le x \le 1$The analytical solution is obtained as $y(x) = \frac{7}{2}sinx + cosx - \frac{x}{2}cosx$

Table 3. Results of problem 3, for $h = 0.01$

Xn	Numerical Solution	Exact Solution	Absolute Error
0.00	1.0000000000	1.0000000000	0.0000000000
0.10	1.3029926711	1.2946709154	0.0083217557
0.20	1.5704419739	1.5693821766	0.0010597973
0.30	1.8497036135	1.8463567391	0.0033468744
0.40	2.1049603428	2.0998129933	0.0051473495
0.50	2.3436635731	2.3361763065	0.0074872666
0.60	2.5647387611	2.5539835873	0.0107551737
0.70	2.7684597943	2.7519093271	0.0165504672
0.80	2.9404151201	2.9287703438	0.0116447764
0.90	3.1486013905	3.0835296662	0.0650717243
1.00	3.2667165936	3.2152995998	0.0514169938

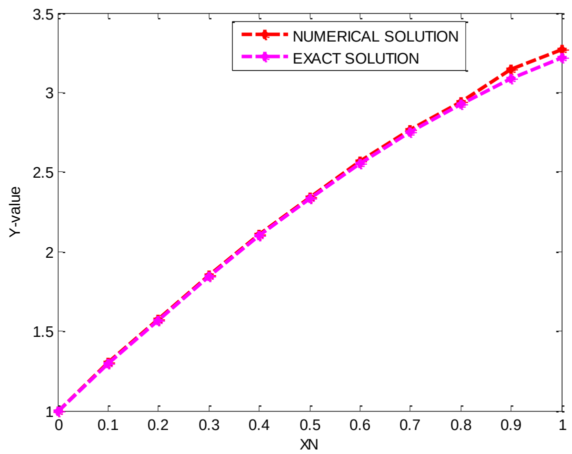

Figure 5. The graph of the solutions to problem 3 on table 3

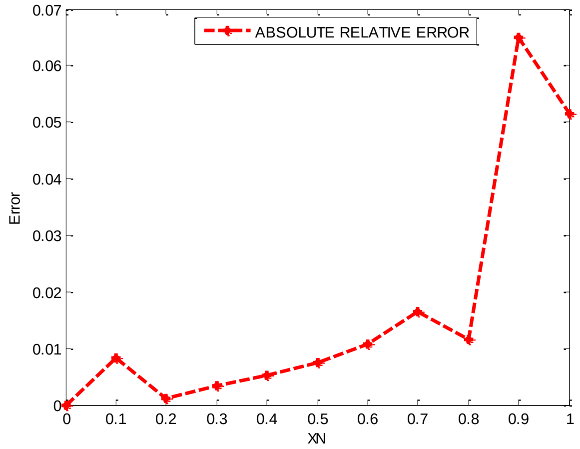

Figure 6. The graph of the absolute error to problem 3

5. Conclusion and Recommendation

In this work, a single-step numerical method has been developed to solve higher-order ordinary differential equation of second-order via a transcendental interpolating function in a range with a basis function consisting of Exponential and Trigonometric functions. The choice of this basis function is to ensure that problems of such possess unique and continuously differentiable solutions. The numerical method tested and found reliable, efficient and less tedious compared to linear multistep methods which require reduction of higher order equations to system of first order equations.

The requisite numerical method properties were obtained in like convergence, consistence, and stable. Three illustrative examples were solved to test the performance of the algorithms in terms of the absolute relative errors computed with the use of MATLAB codes. It was observed that the three examples both homogeneous and non-homogeneous second order special class of equations approximate solutions coincide with the exact solution as shown in the graphs (figures 2,3 and 4) as the step-size h is varied. Hence, the proposed numerical method can be considered to be instrument to solve physical problems in mechanics, and engineering.

References

Abolarin, O. E., Ogunwale, B. G., Adebisi, A. F. and Ayinde, S. O., (2020). Derivation and Implememtation of a Three-Step Hybrid Block Algorithm (TSHBA) for Direct Solution of Linear and Nonlinear Second Order ODEs, *FUDMA Journal of Science (FJS). 4*(4), 477-483

Awoyemi, D. O., and Kayode, S. J., (2005). A Maximal Order Collocation Method for Direct Solution of Initial Value Problems of General Second Order Ordinary Differential Equations. *Proceedings of the conference organized by the Natipnal Mathematical Centre,* Abuja.

Dahlquist G. (1956). Convergence and Stability in the Numerical Integration of Ordinary Differentials Equations. *Math. Scand*. *4,* 33-53.

Fatunla, S.O. (1988). Numerical Methods for Initial Value Problems in Ordinary Differential Equations. *Academic Press*, San Diego, USA.

Frank A., and Elliot M. (2009). Calculus, Fifth Edition. Schaum's Outlines series. *The McGraw Hill Companies.* USA.

Jain, M.K., Iyengar, S.R.K. and Jain, R.K. (2012). Numerical Methods, for Scientific and Engineering Computation 6th Edition. *New Age International (P) Ltd., Publishers*. New Delhi, India. ISBN: 978-81-224-3323-4.

Jator, S. N. (2007). A Sixth-Order Linear Multistep Method for the Direct Solution of Second Order ODEs. *International Journal of Pure and Applied Mathematics. 40,* 457-472. USA

Kayode, S. J. and Adeyeye, O. (2011). A 3-Step Hybrid Method for the Direct Solution of Second Order IVPs. *Australian Journal of Basic and Applied Sciences. 5*(12), 2121-2126. Australia.

Lambert, J.D. (1972). Introductory Mathematics for Scientists and Engineers. *John Wiley & Sons Inc*, New York.

Oguunrinde R. B. and Ayinde S O. (2017). A Numerical Integration for Solving First Order Differential Equations using Gompertz Function approach. *American Journal of Computational and Applied Mathematics*. *7*(6), 143-148. USA

Omar, Z., and Raft, A., (2016) New Uniform Order Single Step Hybrid Block Method for solving Second Order ODEs. *International Journal of Applied Engineering Research. 11*(4), 2402-2406. Algeria

Richard B., and Gabriel B. (2006). Differential Equations. Third Edition. *Schaum's Outlines Series. The McGraw Hill Companies*. USA.

The Open University. (2013). MST 224 Mathematical Methods: Second – order differential equations. *Walton Hall, Keynes*, MK 76AA. ISBN: 987 1 7800 7479 5.

Ogunrinde R. B. and Ayinde S O. (2017). A Numerical Integration for Solving First Order Differential Equations using Gompertz Function approach. *American Journal of Computational and Applied Mathematics*. *7*(6), 143-148. USA

Omar, Z., and Raft, A., (2016). New Uniform Order Single Step Hybrid Block Method for solving Second Order ODEs. *International Journal of Applied Engineering Research. 11*(4), 2402-2406. Algeria

Richard B., and Gabriel B. (2006). Differential Equations. Third Edition. *Schaum's Outlines Series. The McGraw Hill Companies*. USA.

The Open University. (2013). MST 224 Mathematical Methods: Second – order differential equations. *Walton Hall, Keynes*, MK 76AA. ISBN: 987 1 7800 7479 5.