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Sensitivity Analysis Among Commodities and Prices: A Utility Maximization Perception

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Abstract

Utility maximization policy is essential for the sustainability of the firms and organizations. In this article, two Lagrangian multipliers are used to perform the research successfully. Moreover, the paper takes attempts to run with four commodities and two constraints, such as budget constraint, and coupon constraint. In the study, both Hessian and Jacobian are used to show the mathematical calculations precisely and concisely. The sensitivity analysis among two commodities and one price unit of commodity are included in this paper for forecasting on the economic model during the utility maximization procedure.

Keywords: lagrange multipliers, sensitivity analysis, utility maximization

1. Introduction

Utility maximization policy is the best way for the sustainability of the organizations. Because, in the society, the individuals want to obtain the highest level of satisfaction from their purchasing goods and utility maximization helps in this regard (Kirsh, 2017). The concept of utility was established in the late 18th century by the English moral philosopher Jeremy Bentham (1748-1832) and English philosopher John Stuart Mill (1806-1873) (Bentham, 1780; Read, 2004). The US economists John V. Baxley and John C. Moorhouse have analyzed an example of utility maximization, subject to a budget constraint (Baxley & Moorhouse, 1984). Utility influences the organizations to take decision for the future optimum industrial productions (Fishburn, 1970).

The Lagrange multiplier method is a very useful and powerful technique in multivariable calculus. In mathematical economics, it acts for transforming a constrained problem to a higher dimensional unconstrained problem (Islam et al., 2009a, b, 2011). Utility maximization helps the organizations to achieve the maximum profit within the fixed budget (Eaton & Lipsey, 1975).

To predict the future productions of the industries, sensitivity analysis is necessary. In the study we have introduced implicit functions with meaningful economic behavior (Mohajan & Mohajan, 2022a). During the mathematical analysis we have used bordered 6×6 Hessian determinants (Mohajan & Mohajan, 2022c). In this study we have considered the optimization problems of economics from a somewhat wider perspective (Islam et al., 2010; Mohajan, 2021b).

2. Literature Review

The literature review section is a scholarly portion of theses, research papers or books. It deals with the existing knowledge and allows the researcher to identify relevant theories, methods, and gaps in the existing research (Creswell, 2007). It is a secondary research source and does not report a new or a coming research work (Gibbs, 2008). Famous economists John V. Baxley and John C. Moorhouse have discussed the utility maximization method using an explicit example of optimization (Baxley & Moorhouse, 1984). Distinguish mathematician Jamal Nazrul Islam and his coauthors have analyzed the utility maximization policy. In their study they have provided reasonable interpretation of the Lagrange multipliers (Islam et al., 2010).

Jannatul Ferdous and Haradhan Kumar Mohajan have shown a profit maximization problem (Ferdous & Mohajan, 2022). Lia Roy and her coworkers have consulted on cost minimization and have shown mathematical formulation in some detail (Roy et al., 2021). Devajit Mohajan and Haradhan Kumar Mohajan have tried to discuss profit maximization by using four variable inputs (Mohajan & Mohajan, 2022a). Later, in two papers they have discussed utility maximization methods elaborately (Mohajan & Mohajan, 2022b, c).

Haradhan Kumar Mohajan has worked on utility maximization of Bangladeshi consumers (Mohajan, 2021a). He and his coauthors have highlighted on optimization problems for the social welfare (Mohajan et al., 2013). Pahlaj Moolio and his coauthors have used a Lagrange multiplier to solve optimization problems elaborately and proficiently (Moolio et al., 2009).

3. Methodology of the Study

Methodology in any creative research is the organized and meaningful procedural works that follow scientific methods efficiently (Kothari, 2008). Research methodology provides the principles to the researchers for organizing, planning, designing and conducting a good research. Therefore, it is the science and philosophy behind all researches (Legesse, 2014). In this paper, we have considered four commodities X_1, X_2, X_3, and X_4, and two Lagrange multipliers λ_1 and λ_2. We have expanded the bordered Hessian, and have applied the implicit-function theorem. First, we have studied the consumption of commodity α_1 when its price p_1 increases. Later, we have analyzed the consumption of commodity α_2 when the price p_1 of commodity α_1 increases. Finally, we have provided the meaningful interpretation of the desire results.

In the study we have tried our best to maintain the reliability and validity. We have tried to maintain the ethical credibility by citing references properly both in the text and reference list (Mohajan, 2017b, 2021c). In this study we have used both published and unpublished secondary data sources of optimization (Islam et al., 2012a, b; Mohajan, 2018, 2020). We have reviewed the journal articles, conference papers, published books and handbooks, internet, websites, etc. during the preparation of this paper (Chowdhury et al., 2013; Mohajan, 2011, 2012a, b, 2014, 2022a, b).

4. Objective of the Study

The main objective of this paper is to analyze sensitivity of the commodities with respect to prices during the operation of utility maximization of a firm. The other minor objectives are as follows:

- to use the implicit function theorem of the multivariate calculus, and
- to present mathematical calculations in some details.

5. An Economic Model

Let us consider an economic world where there are only four commodities that are X_1, X_2, X_3, and X_4 (Moolio et al., 2009; Roy et al., 2021). Let a consumer wants to purchase only α_1, α_2, α_3, and α_4 amounts from the four commodities X_1, X_2, X_3, and X_4, respectively. The utility function on these four commodities is given by (Mohajan, 2022c; Mohajan & Mohajan, 2022b),

$$v(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\alpha_1\alpha_2\alpha_3\alpha_4. \tag{1}$$

The budget constraint of the consumer can be represented as,

$$B = p_1\alpha_1 + p_2\alpha_2 + p_3\alpha_3 + p_4\alpha_4 \tag{2}$$

where p_1, p_2, p_3, and p_4 are the prices of per unit of commodities α_1, α_2, α_3, and α_4, respectively. Now the coupon constraint is given by,

$$K = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 \tag{3}$$

where c_1, c_2, c_3, and c_4 are the coupons necessary to purchase a unit of commodity of α_1, α_2, α_3, and α_4, respectively.

Using (1), (2), and (3) we can express Lagrangian function $v(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\lambda_1,\lambda_2)$ as (Mohajan, 2017a; Ferdous & Mohajan, 2022),

$$v(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\lambda_1,\lambda_2)=\alpha_1\alpha_2\alpha_3\alpha_4+\lambda_1(B-p_1\alpha_1-p_2\alpha_2-p_3\alpha_3-p_4\alpha_4)$$

$$+\lambda_2(K-c_1\alpha_1-c_2\alpha_2-c_3\alpha_3-c_4\alpha_4). \tag{4}$$

Lagrangian function (4) is a 6-dimensional unconstrained problem that maximizes utility functions; where λ_1 and λ_2 are two Lagrange multipliers that are used as devices of mathematical procedures.

Now we consider the bordered Hessian (Roy et al., 2021; Mohajan & Mohajan, 2022a),

$$|H|=\begin{vmatrix} 0 & 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -K_1 & -K_2 & -K_3 & -K_4 \\ -B_1 & -K_1 & v_{11} & v_{12} & v_{13} & v_{14} \\ -B_2 & -K_2 & v_{21} & v_{22} & v_{23} & v_{24} \\ -B_3 & -K_3 & v_{31} & v_{32} & v_{33} & v_{34} \\ -B_4 & -K_4 & v_{41} & v_{42} & v_{43} & v_{44} \end{vmatrix}. \tag{5}$$

Now taking first and second order and cross-partial derivatives in (4) we obtain (Mohajan & Mohajan, 2022c);

$$B_1=p_1,\ B_2=p_2,\ B_3=p_3,\ B_4=p_4.$$

$$K_1=c_1,\ K_2=c_2,\ K_3=c_3,\ K_4=c_4. \tag{6}$$

$$v_{11}=0,\ v_{12}=v_{21}=\alpha_3\alpha_4,\ v_{13}=v_{31}=\alpha_2\alpha_4,$$

$$v_{14}=v_{41}=\alpha_2\alpha_3,\ v_{22}=0,\ v_{23}=v_{32}=\alpha_1\alpha_4,$$

$$v_{24}=v_{42}=\alpha_1\alpha_3,\ v_{33}=0,\ v_{34}=v_{43}=\alpha_1\alpha_2,\ v_{44}=0. \tag{7}$$

We use $p_1=p_3$ and $p_2=p_4$, i.e., a pair of prices are same, and $c_1=c_3$ and $c_2=c_4$, i.e., a pair of coupon numbers are same. Now we consider that every term contains $p_1p_2c_1c_2$, i.e., we use $p_1^2=p_2^2=p_3^2=p_3^2=p_1p_2$ and $c_1^2=c_2^2=c_3^2=c_3^2=c_1c_2$, where the terms contain square, then (5) becomes (Mohajan & Mohajan, 2022b);

$$|H|=-2p_1p_2c_1c_2<0. \tag{8}$$

We can determine Lagrange multiplier $\lambda_1>0$ as,

$$\lambda_1=\alpha_3\alpha_4\frac{\alpha_2c_2-\alpha_1c_1}{c_2p_1-c_1p_2}. \tag{9}$$

For α_1, α_2, α_3, α_4, λ_1, and λ_2 in terms of p_1, p_2, p_3, p_4, c_1, c_2, c_3, c_4, B, and K we can calculate the sixty partial derivatives, such as $\frac{\partial \lambda_1}{\partial p_1}$, $\frac{\partial \lambda_2}{\partial p_1}$, …, $\frac{\partial \lambda_1}{\partial c_1}$, $\frac{\partial \lambda_2}{\partial c_1}$, …, $\frac{\partial \alpha_1}{\partial p_1}$,…, $\frac{\partial \alpha_1}{\partial c_1}$,…, $\frac{\partial \lambda_1}{\partial B}$,…, $\frac{\partial \lambda_1}{\partial K}$, etc. (Islam et al., 2010). Now we consider Jacobian matrix (Mohajan & Mohajan, 2022a);

$$J = \begin{bmatrix} 0 & 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -K_1 & -K_2 & -K_3 & -K_4 \\ -B_1 & -K_1 & v_{11} & v_{12} & v_{13} & v_{14} \\ -B_2 & -K_2 & v_{21} & v_{22} & v_{23} & v_{24} \\ -B_3 & -K_3 & v_{31} & v_{32} & v_{33} & v_{34} \\ -B_4 & -K_4 & v_{41} & v_{42} & v_{43} & v_{44} \end{bmatrix} \quad (10)$$

is non-singular at the optimum point $\left(\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*, \lambda_1^*, \lambda_2^*\right)$. Since the second order conditions have been satisfied, so the determinant of (10) does not vanish at the optimum, i.e., $|J| = |H|$; and we apply the implicit-function theorem. We have total 16 variables in our study, such as λ_1, λ_2 , $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ p_1, p_2, p_3, p_4, c_1, c_2, c_3, c_4, B, and K. By the implicit function theorem, we can write (Moolio et al., 2009; Islam et al., 2011),

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \mathbf{G}\left(p_1, p_2, p_3, p_4, c_1, c_2, c_3, c_4, B, K\right). \quad (11)$$

Now the 6×10 Jacobian matrix for $\mathbf{G}$, regarded as J_G is given by (Mohajan, 2021a),

$$J_G = \begin{bmatrix} \frac{\partial \lambda_1}{\partial p_1} & \frac{\partial \lambda_1}{\partial p_2} & \frac{\partial \lambda_1}{\partial p_3} & \frac{\partial \lambda_1}{\partial p_4} & \frac{\partial \lambda_1}{\partial c_1} & \frac{\partial \lambda_1}{\partial c_2} & \frac{\partial \lambda_1}{\partial c_3} & \frac{\partial \lambda_1}{\partial c_4} & \frac{\partial \lambda_1}{\partial B} & \frac{\partial \lambda_1}{\partial K} \\ \frac{\partial \lambda_2}{\partial p_1} & \frac{\partial \lambda_2}{\partial p_2} & \frac{\partial \lambda_2}{\partial p_3} & \frac{\partial \lambda_2}{\partial p_4} & \frac{\partial \lambda_2}{\partial c_1} & \frac{\partial \lambda_2}{\partial c_2} & \frac{\partial \lambda_2}{\partial c_3} & \frac{\partial \lambda_2}{\partial c_4} & \frac{\partial \lambda_2}{\partial B} & \frac{\partial \lambda_2}{\partial K} \\ \frac{\partial \alpha_1}{\partial p_1} & \frac{\partial \alpha_1}{\partial p_2} & \frac{\partial \alpha_1}{\partial p_3} & \frac{\partial \alpha_1}{\partial p_4} & \frac{\partial \alpha_1}{\partial c_1} & \frac{\partial \alpha_1}{\partial c_2} & \frac{\partial \alpha_1}{\partial c_3} & \frac{\partial \alpha_1}{\partial c_4} & \frac{\partial \alpha_1}{\partial B} & \frac{\partial \alpha_1}{\partial K} \\ \frac{\partial \alpha_2}{\partial p_1} & \frac{\partial \alpha_2}{\partial p_2} & \frac{\partial \alpha_2}{\partial p_3} & \frac{\partial \alpha_2}{\partial p_4} & \frac{\partial \alpha_2}{\partial c_1} & \frac{\partial \alpha_2}{\partial c_2} & \frac{\partial \alpha_2}{\partial c_3} & \frac{\partial \alpha_2}{\partial c_4} & \frac{\partial \alpha_2}{\partial B} & \frac{\partial \alpha_2}{\partial K} \\ \frac{\partial \alpha_3}{\partial p_1} & \frac{\partial \alpha_3}{\partial p_2} & \frac{\partial \alpha_3}{\partial p_3} & \frac{\partial \alpha_3}{\partial p_4} & \frac{\partial \alpha_3}{\partial c_1} & \frac{\partial \alpha_3}{\partial c_2} & \frac{\partial \alpha_3}{\partial c_3} & \frac{\partial \alpha_3}{\partial c_4} & \frac{\partial \alpha_3}{\partial B} & \frac{\partial \alpha_3}{\partial K} \\ \frac{\partial \alpha_4}{\partial p_1} & \frac{\partial \alpha_4}{\partial p_2} & \frac{\partial \alpha_4}{\partial p_3} & \frac{\partial \alpha_4}{\partial p_4} & \frac{\partial \alpha_4}{\partial c_1} & \frac{\partial \alpha_4}{\partial c_2} & \frac{\partial \alpha_4}{\partial c_3} & \frac{\partial \alpha_4}{\partial c_4} & \frac{\partial \alpha_4}{\partial B} & \frac{\partial \alpha_4}{\partial K} \end{bmatrix}. \quad (12)$$

$$=-J^{-1}\begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & 0 & 1 \\ -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 \end{bmatrix}. \quad (13)$$

The inverse of Jacobian is, $J^{-1}=\frac{1}{|J|}C^T$, where $C=\left(C_{ij}\right)$, the matrix of cofactors of J, and T indicates transpose, then (13) becomes (Mohajan, 2017a; Islam et al., 2011),

$$J_G=-\frac{1}{|J|}C^T\begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & 0 & 1 \\ -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 \end{bmatrix}. \quad (14)$$

Now 6×6 transpose matrix C^T can be represented by,

$$C^T=\begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} & C_{51} & C_{61} \\ C_{12} & C_{22} & C_{32} & C_{42} & C_{52} & C_{62} \\ C_{13} & C_{23} & C_{33} & C_{43} & C_{53} & C_{63} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{54} & C_{64} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{65} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}. \quad (15)$$

Using (15) we can write (12) as 6×10 matrix,

$$J_G=-\frac{1}{|J|}\begin{bmatrix} -\alpha_1C_{11}-\lambda_1C_{31} & -\alpha_2C_{11}-\lambda_1C_{41} & -\alpha_3C_{11}-\lambda_1C_{51} & -\alpha_4C_{11}-\lambda_1C_{61} & -\alpha_1C_{21}-\lambda_1C_{31} & -\alpha_2C_{21}-\lambda_2C_{41} & -\alpha_3C_{21}-\lambda_2C_{51} & -\alpha_4C_{21}-\lambda_2C_{61} & C_{11} & C_{21} \\ -\alpha_1C_{12}-\lambda_1C_{32} & -\alpha_2C_{12}-\lambda_1C_{42} & -\alpha_3C_{12}-\lambda_1C_{52} & -\alpha_4C_{12}-\lambda_1C_{62} & -\alpha_1C_{22}-\lambda_1C_{32} & -\alpha_2C_{22}-\lambda_2C_{42} & -\alpha_3C_{32}-\lambda_2C_{52} & -\alpha_4C_{22}-\lambda_2C_{62} & C_{12} & C_{22} \\ -\alpha_1C_{13}-\lambda_1C_{33} & -\alpha_2C_{13}-\lambda_1C_{43} & -\alpha_3C_{13}-\lambda_1C_{53} & -\alpha_4C_{13}-\lambda_1C_{63} & -\alpha_1C_{23}-\lambda_1C_{33} & -\alpha_2C_{23}-\lambda_2C_{43} & -\alpha_3C_{33}-\lambda_2C_{53} & -\alpha_4C_{23}-\lambda_2C_{63} & C_{13} & C_{23} \\ -\alpha_1C_{14}-\lambda_1C_{34} & -\alpha_2C_{14}-\lambda_1C_{44} & -\alpha_3C_{14}-\lambda_1C_{54} & -\alpha_4C_{14}-\lambda_1C_{64} & -\alpha_1C_{24}-\lambda_1C_{34} & -\alpha_2C_{24}-\lambda_2C_{44} & -\alpha_3C_{34}-\lambda_2C_{54} & -\alpha_4C_{24}-\lambda_2C_{64} & C_{14} & C_{24} \\ -\alpha_1C_{15}-\lambda_1C_{35} & -\alpha_2C_{15}-\lambda_1C_{45} & -\alpha_3C_{15}-\lambda_1C_{55} & -\alpha_4C_{15}-\lambda_1C_{65} & -\alpha_1C_{25}-\lambda_1C_{35} & -\alpha_2C_{25}-\lambda_2C_{45} & -\alpha_3C_{35}-\lambda_2C_{55} & -\alpha_4C_{25}-\lambda_2C_{65} & C_{15} & C_{25} \\ -\alpha_1C_{16}-\lambda_1C_{36} & -\alpha_2C_{16}-\lambda_1C_{46} & -\alpha_3C_{16}-\lambda_1C_{56} & -\alpha_4C_{16}-\lambda_1C_{66} & -\alpha_1C_{26}-\lambda_1C_{36} & -\alpha_2C_{26}-\lambda_2C_{46} & -\alpha_3C_{36}-\lambda_2C_{56} & -\alpha_4C_{26}-\lambda_2C_{66} & C_{16} & C_{26} \end{bmatrix}. \quad (16)$$

Now we analyze the nature of consumption of commodity α_1 when its price increases. Taking T_{31}, (i.e., term of 3rd row and 1st column) from both sides of (16) we get (Islam et al., 2011; Mohajan & Mohajan, 2022c),

$$\frac{\partial \alpha_1}{\partial p_1} = -\frac{1}{|J|}\left[-\alpha_1 C_{13} - \lambda_1 C_{33}\right]$$

$$= \frac{1}{|J|}\left[\alpha_1 C_{13} + \lambda_1 C_{33}\right]$$

$$= \frac{\alpha_1}{|J|}\left[C_{13}\right] + \frac{\lambda_1}{|J|}\left[C_{33}\right]$$

$$= \frac{\alpha_1}{|J|}\text{Cofactor of } C_{13} + \frac{\lambda_1}{|J|}\text{Cofactor of } C_{33}$$

$$= \frac{\alpha_1}{|J|}\begin{vmatrix} 0 & 0 & -K_2 & -K_3 & -K_4 \\ -B_1 & -K_1 & v_{12} & v_{13} & v_{14} \\ -B_2 & -K_2 & v_{22} & v_{23} & v_{24} \\ -B_3 & -K_3 & v_{32} & v_{33} & v_{34} \\ -B_4 & -K_4 & v_{42} & v_{43} & v_{44} \end{vmatrix} + \frac{\lambda_1}{|J|}\begin{vmatrix} 0 & 0 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -K_2 & -K_3 & -K_4 \\ -B_2 & -K_2 & v_{22} & v_{23} & v_{24} \\ -B_3 & -K_3 & v_{32} & v_{33} & v_{34} \\ -B_4 & -K_4 & v_{42} & v_{43} & v_{44} \end{vmatrix}$$

$$= \frac{\alpha_1}{|J|}\left\{-K_2\begin{vmatrix} -B_1 & -K_1 & v_{13} & v_{14} \\ -B_2 & -K_2 & v_{23} & v_{24} \\ -B_3 & -K_3 & v_{33} & v_{34} \\ -B_4 & -K_4 & v_{43} & v_{44} \end{vmatrix} + K_3\begin{vmatrix} -B_1 & -K_1 & v_{12} & v_{14} \\ -B_2 & -K_2 & v_{22} & v_{24} \\ -B_3 & -K_3 & v_{32} & v_{34} \\ -B_4 & -K_4 & v_{42} & v_{44} \end{vmatrix}\right.$$

$$\left. - K_4\begin{vmatrix} -B_1 & -K_1 & v_{12} & v_{13} \\ -B_2 & -K_2 & v_{22} & v_{23} \\ -B_3 & -K_3 & v_{32} & v_{33} \\ -B_4 & -K_4 & v_{42} & v_{43} \end{vmatrix}\right\} + \frac{\lambda_1}{|J|}\left\{-B_2\begin{vmatrix} 0 & -B_2 & -B_3 & -B_4 \\ 0 & -K_2 & -K_3 & -K_4 \\ -K_3 & v_{32} & v_{33} & v_{34} \\ -K_4 & v_{42} & v_{43} & v_{44} \end{vmatrix}\right.$$

$$\left. + B_3\begin{vmatrix} 0 & -B_2 & -B_3 & -B_4 \\ 0 & -K_2 & -K_3 & -K_4 \\ -K_2 & v_{22} & v_{23} & v_{24} \\ -K_4 & v_{42} & v_{43} & v_{44} \end{vmatrix} - B_4\begin{vmatrix} 0 & -B_2 & -B_3 & -B_4 \\ 0 & -K_2 & -K_3 & -K_4 \\ -K_2 & v_{22} & v_{23} & v_{24} \\ -K_3 & v_{32} & v_{33} & v_{34} \end{vmatrix}\right\}$$

$$= \frac{\alpha_1}{|J|}\left[-K_2\left\{-B_1\begin{vmatrix} -K_2 & v_{23} & v_{24} \\ -K_3 & v_{33} & v_{34} \\ -K_4 & v_{43} & v_{44} \end{vmatrix} + K_1\begin{vmatrix} -B_2 & v_{23} & v_{24} \\ -B_3 & v_{33} & v_{34} \\ -B_4 & v_{43} & v_{44} \end{vmatrix} + v_{13}\begin{vmatrix} -B_2 & -K_2 & v_{24} \\ -B_3 & -K_3 & v_{34} \\ -B_4 & -K_4 & v_{44} \end{vmatrix}\right.\right.$$

$$\left. - v_{14}\begin{vmatrix} -B_2 & -K_2 & v_{23} \\ -B_3 & -K_3 & v_{33} \\ -B_4 & -K_4 & v_{43} \end{vmatrix}\right\} + K_3\left\{-B_1\begin{vmatrix} -K_2 & v_{22} & v_{24} \\ -K_3 & v_{32} & v_{34} \\ -K_4 & v_{42} & v_{44} \end{vmatrix} + K_1\begin{vmatrix} -B_2 & v_{22} & v_{24} \\ -B_3 & v_{32} & v_{34} \\ -B_4 & v_{42} & v_{44} \end{vmatrix} + v_{12}\begin{vmatrix} -B_2 & -K_2 & v_{24} \\ -B_3 & -K_3 & v_{34} \\ -B_4 & -K_4 & v_{44} \end{vmatrix}\right.$$

$$-v_{14}\begin{vmatrix}-B_2 & -K_2 & v_{22}\\ -B_3 & -K_3 & v_{32}\\ -B_4 & -K_4 & v_{42}\end{vmatrix}\Bigg\} \qquad -K_4\Bigg\{-B_1\begin{vmatrix}-K_2 & v_{22} & v_{23}\\ -K_3 & v_{32} & v_{33}\\ -K_4 & v_{42} & v_{43}\end{vmatrix} \qquad +K_1\begin{vmatrix}-B_2 & v_{22} & v_{23}\\ -B_3 & v_{32} & v_{33}\\ -B_4 & v_{42} & v_{43}\end{vmatrix}$$

$$+v_{12}\begin{vmatrix}-B_2 & -K_2 & v_{23}\\ -B_3 & -K_3 & v_{33}\\ -B_4 & -K_4 & v_{43}\end{vmatrix} \qquad -v_{13}\begin{vmatrix}-B_2 & -K_2 & v_{22}\\ -B_3 & -K_3 & v_{32}\\ -B_4 & -K_4 & v_{42}\end{vmatrix}\Bigg\} \qquad +\frac{\lambda_1}{|J|}\Bigg[-B_2\Bigg\{-K_3\begin{vmatrix}-B_2 & -B_3 & -B_4\\ -K_2 & -K_3 & -K_4\\ v_{42} & v_{43} & v_{44}\end{vmatrix}$$

$$+K_4\begin{vmatrix}-B_2 & -B_3 & -B_4\\ -K_2 & -K_3 & -K_4\\ v_{32} & v_{33} & v_{34}\end{vmatrix}\Bigg\} \qquad +B_3\Bigg\{-K_2\begin{vmatrix}-B_2 & -B_3 & -B_4\\ -K_2 & -K_3 & -K_4\\ v_{42} & v_{43} & v_{44}\end{vmatrix} \qquad +K_4\begin{vmatrix}-B_2 & -B_3 & -B_4\\ -K_2 & -K_3 & -K_4\\ v_{22} & v_{23} & v_{24}\end{vmatrix}\Bigg\}$$

$$-B_4\Bigg\{-K_2\begin{vmatrix}-B_2 & -B_3 & -B_4\\ -K_2 & -K_3 & -K_4\\ v_{32} & v_{33} & v_{34}\end{vmatrix}+K_3\begin{vmatrix}-B_2 & -B_3 & -B_4\\ -K_2 & -K_3 & -K_4\\ v_{32} & v_{33} & v_{34}\end{vmatrix}\Bigg\}\Bigg]$$

$$=\frac{\alpha_1}{|J|}\big[B_1K_2^2v_{34}^2 \quad -B_1K_2K_4v_{23}v_{34} \quad -B_1K_2K_3v_{24}v_{34} \quad -B_2K_1K_2v_{34}^2 \quad +B_4K_1K_2v_{23}v_{34} \quad -B_3K_1K_2v_{24}v_{34}$$

$$+B_2K_2K_4v_{13}v_{34} \quad -B_4K_2^2v_{13}v_{34} \quad -B_3K_2K_4v_{13}v_{24} \quad +B_4K_2K_3v_{13}v_{24} \quad +B_2K_2K_3v_{14}v_{34} \quad -B_3K_2^2v_{14}v_{34}$$

$$-B_3K_2K_4v_{14}v_{23} \quad +B_4K_2K_3v_{14}v_{23} \quad -B_1K_2K_3v_{24}v_{34} \quad +B_1K_3^2v_{24}^2 \quad -B_1K_3K_4v_{24}^2 \quad +B_2K_1K_3v_{24}v_{34}$$

$$-B_3K_1K_3v_{24}^2$$

$$+B_4K_1K_3v_{23}v_{24}-B_2K_3K_4v_{12}v_{34}+B_4K_2K_3v_{12}v_{34}+B_3K_3K_4v_{12}v_{24}-B_4K_3^2K_3v_{12}v_{24}-B_2K_3^2K_3v_{14}v_{24}$$

$$+B_2K_3K_4v_{14}v_{23} \quad +B_3K_2K_3v_{14}v_{24} \quad -B_4K_2K_3v_{14}v_{23} \quad -B_1K_2K_4v_{23}v_{34} \quad -B_1K_3K_4v_{23}v_{24} \quad +B_1K_4^2v_{23}^2$$

$$+B_2K_1K_4v_{23}v_{34} \quad -B_4K_1K_4v_{23}^2 \quad -B_2K_3K_4v_{12}v_{34} \quad +B_3K_2K_4v_{12}v_{34} \quad -B_3K_4^2v_{12}v_{23} \quad +B_4K_3K_4v_{12}v_{23}$$

$$+B_4K_3K_4v_{12}v_{23}+B_2K_3K_4v_{13}v_{24}-B_2K_4^2v_{13}v_{23}-B_3K_2K_4v_{13}v_{24}+B_4K_2K_4v_{13}v_{23}\big]+\frac{\lambda_1}{|J|}\big[-B_2^2K_3K_4v_{34}$$

$$+B_2B_3K_3K_4v_{24} \quad +B_2B_4K_2K_3v_{34} \quad -B_2B_4K_3^2v_{24} \quad -B_2^2K_3K_4v_{34} \quad +B_2B_3K_2K_4v_{34} \quad -B_2B_3K_4^2v_{23}$$

$$+B_2B_4K_3K_4v_{23} \quad +B_2B_3K_2K_4v_{34} \quad -B_3^2K_2K_4v_{24} \quad -B_3B_4K_2^2v_{34} \quad +B_3B_4K_2K_3v_{24} \quad +B_2B_3K_3K_4v_{24}$$

$$-B_2B_3K_4^2v_{23} \quad -B_3^2K_2K_4v_{24} \quad +B_3B_4K_2K_4v_{23} \quad +B_2B_4K_2K_3v_{34} \quad -B_3B_4K_2^2v_{34} \quad +B_3B_4K_2K_4v_{23}$$

$$-B_4^2K_2K_3v_{23} \qquad -B_2B_4K_3^2v_{34} \qquad +B_3B_4K_2K_3v_{34} \qquad -B_3B_4K_3K_4v_{23} \qquad +B_4^2K_3^2v_{23}\big]$$

$$=\frac{\alpha_1}{|J|}\big[p_1c_2^2\alpha_1^2\alpha_2^2 \quad -p_1c_2c_3\alpha_1^2\alpha_2\alpha_3-p_1c_2c_4\alpha_1^2\alpha_2\alpha_4-p_2c_1c_2\alpha_1^2\alpha_2^2+p_4c_1c_2\alpha_1^2\alpha_2\alpha_4-p_3c_1c_2\alpha_1^2\alpha_2\alpha_3$$

$$+ p_2c_2c_4\alpha_1\alpha_2^2\alpha_4 \quad - p_4c_2^2\alpha_1\alpha_2^2\alpha_4 \quad - p_3c_2c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_4c_2c_3\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_2c_2c_3\alpha_1\alpha_2^2\alpha_3$$

$$- p_3c_2^2\alpha_1\alpha_2^2\alpha_3 \; - p_3c_2c_4\alpha_1\alpha_2\alpha_3\alpha_4 \; + p_4c_2c_3\alpha_1\alpha_2\alpha_3\alpha_4 \; - p_1c_2c_3\alpha_1^2\alpha_2\alpha_3 \; + p_1c_3^2\alpha_1^2\alpha_3^2 \; - p_1c_3c_4\alpha_1^2\alpha_3^2$$

$$+ p_2c_1c_3\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_3c_1c_3\alpha_1^2\alpha_3^2 \quad + p_4c_1c_3\alpha_1^2\alpha_3\alpha_4 \quad - p_2c_3c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_4c_2c_3\alpha_1\alpha_2\alpha_3\alpha_4$$

$$+ p_3c_3c_4\alpha_1\alpha_3^2\alpha_4 \; - p_4c_3^2\alpha_1\alpha_3^2\alpha_4 \; - p_2c_3^2\alpha_1\alpha_2\alpha_3^2 \; - p_3c_1c_3\alpha_1^2\alpha_3^2 \; + p_3c_2c_3\alpha_1\alpha_2\alpha_3^2 \; - p_4c_2c_3\alpha_1\alpha_2\alpha_3\alpha_4$$

$$- p_1c_2c_4\alpha_1^2\alpha_2\alpha_4 \; - p_1c_3c_4\alpha_1^2\alpha_3\alpha_4 \; + p_1c_4^2\alpha_1^2\alpha_4^2 \; + p_1c_3c_4\alpha_1^2\alpha_3\alpha_4 \; + p_3c_1c_4\alpha_1^2\alpha_3\alpha_4 \; - p_4c_1c_4\alpha_1^2\alpha_4^2$$

$$- p_2c_3c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_3c_2c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_3c_4^2\alpha_1\alpha_3\alpha_4^2 \quad + p_4c_3c_4\alpha_1\alpha_3\alpha_4^2 \quad + p_4c_3c_4\alpha_1\alpha_3\alpha_4^2$$

$$+ p_2c_3c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_2c_4^2\alpha_1\alpha_2\alpha_4^2 \quad - p_3c_2c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_4c_2c_4\alpha_1\alpha_3\alpha_4^2 \Big] \; + \frac{\lambda_1}{|J|} \; \Big[- p_2^2c_3c_4\alpha_1\alpha_2$$

$$+ p_2p_3c_3c_4\alpha_1\alpha_3 \; + p_2p_4c_2c_3\alpha_1\alpha_2 \; - p_2p_4c_3^2\alpha_1\alpha_3 \; - p_2^2c_3c_4\alpha_1\alpha_2 \; + p_2p_3c_2c_4\alpha_1\alpha_2 \; - p_2p_3c_4^2\alpha_1\alpha_4$$

$$+ p_2p_4c_3c_4\alpha_1\alpha_4 \; + p_2p_3c_2c_4\alpha_1\alpha_2 \; - p_3^2c_2c_4\alpha_1\alpha_3 \; - p_2p_4c_2^2\alpha_1\alpha_2 \; + p_3p_4c_2c_3\alpha_1\alpha_3 \; + p_2p_3c_3c_4\alpha_1\alpha_3$$

$$- p_2p_3c_4^2\alpha_1\alpha_4 \; - p_3^2c_2c_4\alpha_1\alpha_3 \; + p_3p_4c_2c_4\alpha_1\alpha_4 \; + p_2p_4c_2c_3\alpha_1\alpha_2 \; - p_3p_4c_2^2\alpha_1\alpha_2 \; + p_3p_4c_2c_4\alpha_1\alpha_4$$

$$- p_4^2c_2c_3\alpha_1\alpha_4 - p_2p_4c_3^2\alpha_1\alpha_2 + p_3p_4c_2c_3\alpha_1\alpha_2 - p_3p_4c_3c_4\alpha_1\alpha_4 + p_4^2c_3^2\alpha_1\alpha_4 \Big]$$

$$= \frac{\alpha_1}{|J|} \quad \Big[\left(p_1c_2^2 - p_2c_1c_2\right)\alpha_1^2\alpha_2^2 \quad + \left(p_1c_3^2 - p_1c_3c_4 - 2p_3c_1c_3\right)\alpha_1^2\alpha_3^2 \quad + \left(- p_4c_1c_4 + p_1c_4^2\right)\alpha_1^2\alpha_4^2$$

$$- p_2c_4^2\alpha_1\alpha_2\alpha_4^2 \quad + \left(- p_3c_1c_2 - 2p_1c_2c_3\right)\alpha_1^2\alpha_2\alpha_3 \quad + \left(p_4c_1c_2 - 2p_1c_2c_4\right)\alpha_1^2\alpha_2\alpha_4$$

$$+ \left(p_2c_2c_4 - p_4c_2^2\right)\alpha_1\alpha_2^2\alpha_4 \quad + \left(p_2c_2c_3 - p_3c_2^2\right)\alpha_1\alpha_2^2\alpha_3$$

$$+ \left(p_4c_1c_3 + p_3c_1c_4\right)\alpha_1^2\alpha_3\alpha_4 \quad + \left(p_3c_3c_4 - p_4c_3^2\right)\alpha_1\alpha_3^2\alpha_4 \quad + \left(p_3c_2c_3 - p_2c_3^2\right)\alpha_1\alpha_2\alpha_3^2$$

$$+ \left(p_2c_1c_3 - p_2c_3c_4 - 2p_3c_2c_4 + 2p_4c_2c_3\right)\alpha_1\alpha_2\alpha_3\alpha_4 + \left(- p_3c_4^2 + 2p_4c_3c_4 + p_4c_2c_4\right)\alpha_1\alpha_3\alpha_4^2\Big]$$

$$+ \frac{\lambda_1}{|J|}\Big[\left(-2p_2^2c_3c_4 + 2p_2p_3c_2c_4 + 2p_2p_4c_2c_3 - p_2p_4c_2^2 - p_3p_4c_2^2 - p_2p_4c_3^2 + p_3p_4c_2c_3\right)\alpha_1\alpha_2$$

$$+ \left(2p_2p_3c_3c_4 - p_2p_4c_3^2 - 2p_3^2c_2c_4 + p_3p_4c_2c_3\right)\alpha_1\alpha_3$$

$$+ \left(- p_2p_3c_4^2 + p_2p_4c_3c_4 - p_2p_3c_4^2 + 2p_3p_4c_2c_4 - p_4^2c_2c_3 - p_3p_4c_3c_4 + p_4^2c_3^2\right)\alpha_1\alpha_4\Big] \qquad .$$

(17)

Now using $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ in (17) we get,

$$= \frac{1}{|J|}\left[\left(p_1c_2^2 + p_1c_3^2 + p_1c_4^2 + p_4c_1c_2 + 2p_4c_2c_3 + p_2c_1c_3 + p_4c_1c_3 + p_3c_1c_4 + p_2c_2c_3 + p_2c_2c_4\right.\right.$$

$$+ p_3c_2c_3$$

$$\left. + p_3c_3c_4 + p_4c_2c_4 + 2p_4c_3c_4\right) - \left(p_2c_1c_2 + 2p_3c_1c_3 + p_4c_1c_4 + p_4c_2^2 + p_2c_3c_4 + p_3c_1c_2 + 2p_1c_2c_4\right.$$

$$\left.\left. + 2p_3c_2c_4 + 2p_1c_2c_3 + p_1c_3c_4 + p_2c_3^2 + p_4c_3^2 + p_2c_4^2 + p_3c_4^2 + p_3c_2^2\right)\right] + \frac{\lambda_1}{|J|}\left[\left(2p_2p_3c_2c_4\right.\right.$$

$$\left. + 2p_2p_4c_2c_3 + p_3p_4c_2c_3 + 2p_2p_3c_3c_4 + p_3p_4c_2c_3 + p_2p_4c_3c_4 + 2p_3p_4c_2c_4 + p_4^2c_3^2\right) - \left(2p_2^2c_3c_4\right.$$

$$\left.\left. + p_2p_4c_2^2 + p_3p_4c_2^2 + p_2p_4c_3^2 + p_2p_4c_3^2 + 2p_3^2c_2c_4 + p_2p_3c_4^2 + p_2p_3c_4^2 + p_4^2c_2c_3 + p_3p_4c_3c_4\right)\right]. \quad (18)$$

We consider; $p_1 = p_2 = p_3 = p_4 = p$, and $c_1 = c_2 = c_3 = c_4 = c$, then $|J| = -2p^2c^2$ and $\lambda_1 = 1$; (18) gives,

$$\frac{\partial \alpha_1}{\partial p_1} = \frac{-pc^2}{|J|} = \frac{-pc^2}{-2p^2c^2} = \frac{1}{2p} > 0. \quad (19)$$

Relation (19) indicates that if the price of commodity α_1 increases, the level of consumption of α_1 will also increase. In these circumstances we consider that commodity α_1 is a superior good, and it has no other substitutes to get (Islam et al., 2010; Mohajan & Mohajan, 2022c).

We consider $p_3 = p_1$ and $p_4 = p_2$; and $c_3 = c_1$, and $c_4 = c_2$, then $|J| = |H| = -2p_1p_2c_1c_2$, and from (9) using $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ we get,

$$\lambda_1 = \frac{c_2 - c_1}{c_2p_1 - c_1p_2}. \quad (20)$$

Using the value of λ_1 from (20) in (18) we can write,

$$\frac{\partial \alpha_1}{\partial p_1} = \frac{p_2c_1c_2 + p_2c_1^2 - 4p_1c_2^2 - p_1c_1^2 - p_1c_1c_2}{-2p_1p_2c_1c_2} + \frac{(c_2 - c_1)\left(p_1p_2c_2^2 + 3p_1p_2c_1c_2 - p_2^2c_2^2 - p_2^2c_1^2 - 2p_1^2c_1^2\right)}{-2p_1p_2c_1c_2\left(c_2p_1 - c_1p_2\right)} \quad (21)$$

where $c_2p_1 \neq c_1p_2$. Now let $c_1 = c_2 = c$, then from (21) we get,

$$\frac{\partial \alpha_1}{\partial p_1} = \frac{9p_1 - 2p_2}{2p_1p_2}. \quad (22)$$

If $9p_1 > 2p_2$ then from (22) we can write,

$$\frac{\partial \alpha_1}{\partial p_1} > 0. \quad (23)$$

Which bears the same properties as of the relation (19).

If $2p_2 > 9p_1$ then (22) gives,

$$\frac{\partial \alpha_1}{\partial p_1} < 0. \quad (24)$$

Relation (24) indicates that if the price of commodity α_1 increases, the level of consumption of α_1 will decrease. This situation seems reasonable result in the sense that commodity α_1 has many substitutes; and hence consumers switch to substitutes when price of commodity α_1 goes up.

If $9p_1 = 2p_2$ then from (22) we can write,

$$\frac{\partial \alpha_1}{\partial p_1} = 0. \tag{25}$$

Equation (25) indicates that if the price of commodity α_1 increases, there seems no effect on the level of consumption of goods α_1. It looks like commodity α_1 is a necessity and it has neither complementary nor supplementary goods (Islam et al., 2010).

Now we analyze the nature of consumption of commodity α_2 when the price of α_1 increases. Taking T_{41}, (i.e., term of 4th row and 1st column) from both sides of (14) we get (Mohajan & Mohajan, 2022b),

$$\frac{\partial \alpha_2}{\partial p_1} = -\frac{1}{|J|}\left[-\alpha_1 C_{14} - \lambda_1 C_{34}\right]$$

$$= \frac{1}{|J|}\left[\alpha_1 C_{14} + \lambda_1 C_{34}\right]$$

$$= \frac{\alpha_1}{|J|}\left[C_{14}\right] + \frac{\lambda_1}{|J|}\left[C_{34}\right]$$

$$= \frac{\alpha_1}{|J|}\text{Cofactor of } C_{14} + \frac{\lambda_1}{|J|}\text{Cofactor of } C_{34}$$

$$= -\frac{\alpha_1}{|J|}\begin{vmatrix} 0 & 0 & -K_1 & -K_3 & -K_4 \\ -B_1 & -K_1 & v_{11} & v_{13} & v_{14} \\ -B_2 & -K_2 & v_{21} & v_{23} & v_{24} \\ -B_3 & -K_3 & v_{31} & v_{33} & v_{34} \\ -B_4 & -K_4 & v_{41} & v_{43} & v_{44} \end{vmatrix} - \frac{\lambda_1}{|J|}\begin{vmatrix} 0 & 0 & -B_1 & -B_2 & -B_4 \\ 0 & 0 & -K_1 & -K_2 & -K_4 \\ -B_2 & -K_2 & v_{21} & v_{22} & v_{24} \\ -B_3 & -K_3 & v_{31} & v_{32} & v_{34} \\ -B_4 & -K_4 & v_{41} & v_{42} & v_{44} \end{vmatrix}$$

$$= -\frac{\alpha_1}{|J|}\left\{-K_1\begin{vmatrix} -B_1 & -K_1 & v_{13} & v_{14} \\ -B_2 & -K_2 & v_{23} & v_{24} \\ -B_3 & -K_3 & v_{33} & v_{34} \\ -B_4 & -K_4 & v_{43} & v_{44} \end{vmatrix} + K_3\begin{vmatrix} -B_1 & -K_1 & v_{11} & v_{14} \\ -B_2 & -K_2 & v_{21} & v_{24} \\ -B_3 & -K_3 & v_{31} & v_{34} \\ -B_4 & -K_4 & v_{41} & v_{44} \end{vmatrix} - K_4\begin{vmatrix} -B_1 & -K_1 & v_{11} & v_{13} \\ -B_2 & -K_2 & v_{21} & v_{23} \\ -B_3 & -K_3 & v_{31} & v_{33} \\ -B_4 & -K_4 & v_{41} & v_{43} \end{vmatrix}\right\}$$

$$-\frac{\lambda_1}{|J|}\left\{-B_1\begin{vmatrix} 0 & 0 & -K_2 & -K_4 \\ -B_2 & -K_2 & v_{22} & v_{24} \\ -B_3 & -K_3 & v_{32} & v_{34} \\ -B_4 & -K_4 & v_{42} & v_{44} \end{vmatrix} + B_2\begin{vmatrix} 0 & 0 & -K_1 & -K_4 \\ -B_2 & -K_2 & v_{21} & v_{24} \\ -B_3 & -K_3 & v_{31} & v_{34} \\ -B_4 & -K_4 & v_{41} & v_{44} \end{vmatrix} - B_4\begin{vmatrix} 0 & 0 & -K_1 & -K_2 \\ -B_2 & -K_2 & v_{21} & v_{22} \\ -B_3 & -K_3 & v_{31} & v_{32} \\ -B_4 & -K_4 & v_{41} & v_{42} \end{vmatrix}\right\}$$

$$=-\frac{\alpha_1}{|J|}\left[-K_1\left\{-B_1\begin{vmatrix}-K_2 & v_{23} & v_{24}\\ -K_3 & v_{33} & v_{34}\\ -K_4 & v_{43} & v_{44}\end{vmatrix}+K_1\begin{vmatrix}-B_2 & v_{23} & v_{24}\\ -B_3 & v_{33} & v_{34}\\ -B_4 & v_{43} & v_{44}\end{vmatrix}+v_{13}\begin{vmatrix}-B_2 & -K_2 & v_{24}\\ -B_3 & -K_3 & v_{34}\\ -B_4 & -K_4 & v_{44}\end{vmatrix}\right.\right.$$

$$\left.-v_{14}\begin{vmatrix}-B_2 & -K_2 & v_{23}\\ -B_3 & -K_3 & v_{33}\\ -B_4 & -K_4 & v_{43}\end{vmatrix}\right\}$$

$$+K_3\left\{-B_1\begin{vmatrix}-K_2 & v_{21} & v_{24}\\ -K_3 & v_{31} & v_{34}\\ -K_4 & v_{41} & v_{44}\end{vmatrix}+K_1\begin{vmatrix}-B_2 & -K_2 & v_{24}\\ -B_3 & -K_3 & v_{34}\\ -B_4 & -K_4 & v_{44}\end{vmatrix}-v_{14}\begin{vmatrix}-B_2 & -K_2 & v_{21}\\ -B_3 & -K_3 & v_{31}\\ -B_4 & -K_4 & v_{41}\end{vmatrix}\right\}$$

$$-K_4\left\{-B_1\begin{vmatrix}-K_2 & v_{21} & v_{23}\\ -K_3 & v_{31} & v_{33}\\ -K_4 & v_{41} & v_{43}\end{vmatrix}+K_1\begin{vmatrix}-B_2 & v_{21} & v_{23}\\ -B_3 & v_{31} & v_{33}\\ -B_4 & v_{41} & v_{43}\end{vmatrix}-v_{13}\begin{vmatrix}-B_2 & -K_2 & v_{21}\\ -B_3 & -K_3 & v_{31}\\ -B_4 & -K_4 & v_{41}\end{vmatrix}\right\}$$

$$-\frac{\lambda_1}{|J|}\left[-B_1\left\{-K_2\begin{vmatrix}-B_2 & -K_2 & v_{24}\\ -B_3 & -K_3 & v_{34}\\ -B_4 & -K_4 & v_{44}\end{vmatrix}+K_4\begin{vmatrix}-B_2 & -K_2 & v_{22}\\ -B_3 & -K_3 & v_{32}\\ -B_4 & -K_4 & v_{42}\end{vmatrix}\right\}\right.$$

$$+B_2\left\{-K_1\begin{vmatrix}-B_2 & -K_2 & v_{24}\\ -B_3 & -K_3 & v_{34}\\ -B_4 & -K_4 & v_{44}\end{vmatrix}+K_4\begin{vmatrix}-B_2 & -K_2 & v_{21}\\ -B_3 & -K_3 & v_{31}\\ -B_4 & -K_4 & v_{41}\end{vmatrix}\right\}-B_4\left\{-K_1\begin{vmatrix}-B_2 & -K_2 & v_{22}\\ -B_3 & -K_3 & v_{32}\\ -B_4 & -K_4 & v_{42}\end{vmatrix}\right.$$

$$\left.\left.+K_2\begin{vmatrix}-B_2 & -K_2 & v_{21}\\ -B_3 & -K_3 & v_{31}\\ -B_4 & -K_4 & v_{41}\end{vmatrix}\right\}\right]$$

$$=-\frac{\alpha_1}{|J|}\Big[B_1K_1K_2v_{34}^2 \quad -B_1K_1K_4B_1v_{23}v_{34} \quad -B_1K_1K_3v_{24}v_{34} \quad -B_2K_1^2v_{34}^2 \quad +B_4K_1^2v_{23}v_{34} \quad +B_3K_1^2v_{24}v_{34}$$

$$-B_2K_1K_3v_{13}v_{34} \quad +B_3K_1K_2v_{13}v_{34} \quad -B_3K_1K_4v_{13}v_{24} \quad +B_4K_1K_3v_{13}v_{24} \quad +B_2K_1K_3v_{14}v_{34} \quad -B_3K_1K_2v_{14}v_{34}$$

$$+B_3K_1K_4v_{14}v_{23} \quad -B_4K_1K_3v_{14}v_{23} \quad -B_1K_2K_3v_{14}v_{34} \quad +B_1K_3K_4v_{12}v_{34} \quad +B_1K_3^2v_{14}v_{24} \quad -B_1K_3K_4v_{13}v_{24}$$

$$-B_2K_1K_3^2v_{34} \quad +B_4K_1K_2K_3v_{34} \quad +B_3K_1K_3K_4v_{24} \quad -B_4K_1K_3^2v_{24} \quad +B_2K_3^2v_{14}^2 \quad +B_2K_3K_4v_{13}v_{14}$$

$$-B_4K_2K_3v_{14}v_{34} \quad -B_3K_3K_4v_{12}v_{14} \quad +B_4K_3^2v_{12}v_{14} \quad -B_1K_2K_4v_{13}v_{34} \quad +B_1K_3K_4v_{12}v_{34} \quad -B_1K_3K_4v_{14}v_{23}$$

$$+B_1K_4^2v_{13}v_{23} \quad +B_2K_1K_4v_{13}v_{34} \quad -B_3K_1K_4v_{12}v_{34} \quad +B_3K_1K_4v_{14}v_{23} \quad -B_4K_1K_4v_{13}v_{23} \quad +B_2K_3K_4v_{13}v_{14}$$

$$-B_2K_4^2v_{13}^2 \quad -B_3K_2K_4v_{13}v_{14} \quad +B_4K_2K_4v_{13}^2 \quad +B_3K_4^2v_{12}v_{13} \quad -B_4K_3K_4v_{12}v_{13}\Big] \quad -\frac{\lambda_1}{|J|}\Big[-B_1B_2K_2K_4v_{34}$$

$$+B_1B_4K_2^2v_{34} \quad +B_2B_3K_2K_4v_{24} \quad -B_2B_4K_2K_3v_{24} \quad -B_1B_2K_3K_4v_{24} \quad +B_1B_2K_4^2v_{23} \quad +B_1B_3K_2K_4v_{24}$$

$$-B_1B_4K_2K_4v_{23} + B_2^2K_1K_4v_{34} - B_2B_4K_1K_3v_{34} - B_2B_3K_1K_4v_{24} + B_2B_4K_1K_3v_{24} + B_1B_2K_3K_4v_{14}$$
$$-B_1B_2K_4^2v_{13} + B_1B_3K_2K_4v_{14} - B_1B_4K_2K_4v_{13} - B_1B_3K_4^2v_{12} + B_1B_4K_3K_4v_{12} + B_2B_4K_1K_3v_{24}$$
$$-B_2B_4K_1K_4v_{23} - B_3B_4K_1K_2v_{24} + B_4^2K_1K_2v_{23} - B_2B_4K_2K_3v_{14} + B_2B_4K_2K_4v_{13} - B_3B_4K_2^2v_{14}$$
$$+B_4^2K_2^2v_{13} - B_3B_4K_2K_4v_{12} + B_4^2K_2K_3v_{12}\Big]$$

$$= -\frac{\alpha_1}{|J|}\Big[p_1c_1c_2\alpha_1^2\alpha_2^2 - p_1c_1c_4\alpha_1^2\alpha_2\alpha_4 - p_1c_1c_3\alpha_1^2\alpha_2\alpha_3 - p_2c_1^2\alpha_1^2\alpha_2^2 + p_2c_1^2\alpha_1^2\alpha_2\alpha_4 + p_3c_1^2\alpha_1^2\alpha_2\alpha_3$$
$$-p_2c_1c_3\alpha_1\alpha_2^2\alpha_4 + p_3c_1c_2\alpha_1\alpha_2^2\alpha_4 - p_3c_1c_4\alpha_1\alpha_2\alpha_3\alpha_4 + p_4c_1c_3\alpha_1\alpha_2\alpha_3\alpha_4 + p_2c_1c_3\alpha_1\alpha_2^2\alpha_3$$
$$-p_3c_1c_2\alpha_1\alpha_2^2\alpha_3 + p_3c_1c_4\alpha_1\alpha_2\alpha_3\alpha_4 - p_4c_1c_3\alpha_1\alpha_2\alpha_3\alpha_4 - p_1c_2c_3\alpha_1\alpha_2^2\alpha_3 + p_1c_3c_4\alpha_1\alpha_2\alpha_3\alpha_4$$
$$+p_1c_3^2\alpha_1\alpha_2\alpha_3^2 - p_1c_3c_4\alpha_1\alpha_2\alpha_3\alpha_4 - p_2c_1c_3^2\alpha_1\alpha_2 + p_4c_1c_2c_3\alpha_1\alpha_2 - p_4c_1c_3^2\alpha_1\alpha_3 + p_2c_3^2\alpha_2^2\alpha_3^2$$
$$+p_2c_3c_4\alpha_2^2\alpha_3\alpha_4 - p_4c_2c_3\alpha_1\alpha_2^2\alpha_3 - p_3c_3c_4\alpha_2\alpha_3^2\alpha_4 + p_1c_1c_3\alpha_1\alpha_2\alpha_3\alpha_4 - p_1c_2c_4\alpha_1\alpha_2^2\alpha_4$$
$$-p_1c_3c_4\alpha_1\alpha_2\alpha_3\alpha_4 + p_1c_4^2\alpha_2^2\alpha_4^2 + p_2c_1c_4\alpha_1\alpha_2^2\alpha_4 - p_3c_1c_4\alpha_1\alpha_2\alpha_3\alpha_4 + p_3c_1c_4\alpha_1\alpha_2\alpha_3\alpha_4$$
$$-p_4c_1c_4\alpha_1\alpha_2\alpha_4^2 + p_2c_3c_4\alpha_2^2\alpha_3\alpha_4 - p_2c_4^2\alpha_2^2\alpha_4^2 - p_3c_2c_4\alpha_2^2\alpha_3\alpha_4 + p_4c_2c_4\alpha_2^2\alpha_4^2 + p_3c_4^2\alpha_2\alpha_3\alpha_4^2\Big]$$
$$-\frac{\lambda_1}{|J|}\Big[-p_1p_2c_2c_4\alpha_1\alpha_2 + p_1p_4c_2^2\alpha_1\alpha_2 + p_2p_3c_2c_4\alpha_1\alpha_3 - p_2p_4c_2c_3\alpha_1\alpha_3 - p_1p_2c_3c_4\alpha_1\alpha_3$$
$$+p_1p_2c_4^2\alpha_1\alpha_4 + p_1p_3c_2c_4\alpha_1\alpha_3 - p_1p_4c_2c_4\alpha_1\alpha_4 + p_2^2c_1c_4\alpha_1\alpha_2 - p_2p_4c_1c_3\alpha_1\alpha_2 - p_2p_3c_1c_3\alpha_1\alpha_3$$
$$+p_2p_4c_1c_3\alpha_1\alpha_3 + p_1p_2c_3c_4\alpha_2\alpha_3 - p_1p_2c_4^2\alpha_2\alpha_4 + p_1p_3c_2c_4\alpha_2\alpha_3 - p_1p_4c_2c_4\alpha_2\alpha_4 - p_1p_3c_4^2\alpha_3\alpha_4$$
$$+p_1p_4c_3c_4\alpha_3\alpha_4 + p_2p_4c_1c_3\alpha_1\alpha_3 - p_2p_4c_1c_4\alpha_1\alpha_4 - p_2p_4c_1c_2\alpha_1\alpha_3 + p_4^2c_1c_2\alpha_1\alpha_4 - p_2p_4c_2c_3\alpha_2\alpha_3$$
$$+p_2p_4c_2c_4\alpha_2\alpha_4 - p_3p_4c_2^2\alpha_2\alpha_3 + p_4^2c_2^2\alpha_2\alpha_4 - p_3p_4c_2c_4\alpha_3\alpha_4 + p_4^2c_2c_3\alpha_3\alpha_4\Big]$$

$$= -\frac{\alpha_1}{|J|}\Big[\left(p_1c_1c_2 - p_2c_1^2\right)\alpha_1^2\alpha_2^2 + p_2c_3^2\alpha_2^2\alpha_3^2 + \left(p_1c_4^2 + p_4c_2c_4\right)\alpha_2^2\alpha_4^2 + p_1c_3^2\alpha_1\alpha_2\alpha_3^2 - p_3c_3c_4\alpha_2\alpha_3^2\alpha_4$$
$$+\left(-p_1c_1c_4 - p_1c_1c_3 + p_2c_1^2 + p_3c_1^2\right)\alpha_1^2\alpha_2\alpha_4 + \left(-p_2c_1c_3 + p_3c_1c_2 - p_1c_2c_4 + p_2c_1c_4\right)\alpha_1\alpha_2^2\alpha_4$$
$$+\left(p_2c_1c_3 - p_3c_1c_2 - p_1c_2c_3 - p_4c_2c_3\right)\alpha_1\alpha_2^2\alpha_3 + \left(p_2c_3c_4 - p_3c_2c_4 + p_2c_3c_4\right)\alpha_2^2\alpha_3\alpha_4$$
$$+\left(-p_2c_1c_3^2 + p_4c_1c_2c_3 - p_4c_1c_3^2\right)\alpha_1\alpha_2 + \left(-p_4c_1c_4 - p_2c_4^2 + p_3c_4^2\right)\alpha_2\alpha_3\alpha_4^2\Big]$$

$$-\frac{\lambda_1}{|J|}\Big[\left(-p_1p_2c_2c_4+p_1p_4c_2^2+p_2^2c_1c_4-p_2p_4c_1c_3\right)\alpha_1\alpha_2$$

$$+\left(p_1p_2c_4^2-p_1p_4c_2c_4-p_2p_4c_1c_4+p_4^2c_1c_2\right)\alpha_1\alpha_4$$

$$+\left(p_1p_2c_3c_4+p_1p_3c_2c_4-p_2p_4c_2c_3-p_3p_4c_2^2\right)\alpha_2\alpha_3+\left(-p_1p_2c_4^2-p_1p_4c_2c_4+p_2p_4c_2c_4\right)\alpha_2\alpha_4$$

$$+\left(p_2p_3c_2c_4-p_2p_4c_2c_3-p_1p_2c_3c_4+p_1p_3c_2c_4-p_2p_3c_1c_3+2p_2p_4c_1c_3-p_2p_4c_1c_2\right)\alpha_1\alpha_3$$

$$+\left(-p_1p_3c_4^2+p_1p_4c_3c_4+p_4^2c_2^2-p_3p_4c_2c_4+p_4^2c_2c_3\right)\alpha_3\alpha_4\Big].$$

(26)

Now we use $p_3=p_1$ and $p_4=p_2$ where pair of prices are same, and $c_3=c_1$ and $c_4=c_2$, i.e., two types of coupon numbers are same. We put $\alpha_1=\alpha_2=\alpha_3=\alpha_4=1$ then (26) becomes,

$$\frac{\partial\alpha_2}{\partial p_1}=-\frac{1}{|J|}\big[p_1c_1c_2-p_2c_1^2+p_2c_1^2+p_1c_2^2+p_2c_1c_2-p_1c_1c_2-p_1c_1^2+p_2c_1^2+p_1c_1^2-p_2c_1^2$$

$$+p_1c_1c_2-p_1c_2^2+p_2c_1c_2+p_2c_1^2-p_1c_1c_2-p_1c_1c_2-p_2c_1c_2+p_2c_1c_2-p_1c_2^2+p_2c_1c_2+p_1c_1^2$$

$$-p_1c_1c_2-p_2c_1^3+p_2c_1^2c_2-p_2c_1^3-p_2c_1c_2-p_2c_2^2+p_1c_2^2\big]-\frac{\lambda_1}{|J|}\big[-p_1p_2c_1c_2+p_1p_2c_2^2+p_2^2c_1c_2$$

$$-p_2^2c_1^2+p_1p_2c_1c_2-p_2^2c_1c_2-p_1p_2c_1c_2+p_1^2c_1c_2-p_1p_2c_1^2+2p_2^2c_1^2-p_1p_2c_1c_2+p_1p_2c_2^2-p_1p_2c_2^2$$

$$-p_2^2c_1c_2+p_2^2c_1c_2+p_1p_2c_1c_2+p_1^2c_2^2-p_2^2c_1c_2-p_1p_2c_2^2-p_1p_2c_2^2-p_1p_2c_2^2+p_2^2c_2^2-p_1^2c_2^2$$

$$+p_1p_2c_1c_2+p_2^2c_2^2-p_1p_2c_2^2+p_2^2c_1c_2\big].$$

(27)

Now we use $\lambda_1=1$ and $|J|=-2p_1p_2c_1c_2$ in (27), and then we get,

$$\frac{\partial\alpha_2}{\partial p_1}=\left(2p_1p_2c_1c_2\right)^{-1}\big[p_1c_1^2+2p_2c_1c_2+p_2c_1^2-2p_1c_1c_2+p_2c_1^2c_2-p_2c_2^2+p_1^2c_1c_2-p_1p_2c_1^2+p_2^2c_1^2$$

$$-3p_1p_2c_2^2+2p_2^2c_2^2\big]. \quad (28)$$

Now we use $p_1=p_2=p$ and $c_1=c_2=c$ in (28), and then we get,

$$\frac{\partial\alpha_2}{\partial p_1}=\frac{1}{2p^2c^2}\left(pc^2+pc^3\right)>0. \quad (29)$$

The relation (29) indicates that if the price of the commodity α_1 increases, the level of consumption of α_2 will increase. This situation shows that goods α_1 and α_2 are supplementary; that is, when price of α_1 goes up people switch to its supplementary commodity α_2; for instance, tea and coffee (Islam et al., 2010).

Using $\lambda_1=\dfrac{c_2-c_1}{c_2p_1-c_1p_2}$ and $|J|=-2p_1p_2c_1c_2$ in relation (27) we get,

$$\frac{\partial\alpha_2}{\partial p_1}=\frac{1}{2p_1p_2c_1c_2}\left(p_1c_1^2+2p_2c_1c_2+p_2c_1^2-2p_1c_1c_2\right)$$

$$+\frac{1}{2p_1p_2c_1c_2}\frac{c_2-c_1}{c_2p_1-c_1p_2}\left(p_2c_1^2c_2-p_2c_2^2+p_1^2c_1c_2-p_1p_2c_1^2+p_2^2c_1^2-3p_1p_2c_2^2+2p_2^2c_2^2\right)$$

where $c_2p_1\neq c_1p_2$

$$\frac{\partial\alpha_2}{\partial p_1}=\frac{5p_1p_2c_1c_2^2-p_1^2c_1c_2^2-p_2^2c_1^2c_2-2p_2^2c_1^3+2p_1p_2c_1^2c_2-p_2c_2^3}{-2p_1p_2c_1c_2\left(c_2p_1-c_1p_2\right)}$$

$$+\frac{p_2c_1^2c_2^2-3p_1p_2c_2^3+2p_2^2c_2^3-p_2c_1^3c_2+p_2c_1c_2^2-2p_2^2c_1c_2^2}{-2p_1p_2c_1c_2\left(c_2p_1-c_1p_2\right)} \tag{30}$$

where $c_2p_1\neq c_1p_2$. Now let, $c_1=c_2=c$, and then from (30) we get,

$$\frac{\partial\alpha_2}{\partial p_1}=\frac{p_1^2+5p_2^2-4p_1p_2-2p_2}{p_1p_2\left(p_1-p_2\right)}$$

$$=\frac{\left(p_1-2p_2\right)^2+p_2^2-2p_2}{p_1p_2\left(p_1-p_2\right)} \tag{31}$$

where $p_1\neq p_2$.

If $p_1>p_2>2$ then relation (31) gives,

$$\frac{\partial\alpha_2}{\partial p_1}>0 \tag{32}$$

which has same properties as in relation (29).

If $p_1>p_2$ and $p_2<2$ then relation (31) provides,

$$\frac{\partial\alpha_2}{\partial p_1}<0. \tag{33}$$

Relation (33) indicates that if the price of the commodity α_1 increases, the level of consumption of α_2 will decrease. This situation shows that goods α_1 and α_2 are complementary; that is, when price of α_1 goes up people buy less of it; consequently, level of consumption of α_2 also decreases, as because complementary goods are used together; for instance, lemon and tea (Islam et al., 2010).

If $p_2>p_1$ and $p_2>2$ then relation (31) offers,

$$\frac{\partial\alpha_2}{\partial p_1}<0. \tag{34}$$

The goods behave same as (33).

If $p_2>p_1$ and $p_2<2$ then relation (31) provides,

$$\frac{\partial\alpha_2}{\partial p_1}>0. \tag{35}$$

This shows same properties as in relation (32).

If $p_1>p_2$ and $p_2=2$ we have from relation (31),

$$\frac{\partial \alpha_2}{\partial p_1} = 0. \tag{36}$$

In this case we observe that commodities α_1 and α_2 are non-related goods, for example, text books and apples. Therefore, if the price of the commodity α_1 increases, there seems no effect on the level of the consumption of goods α_2 (Islam et al., 2010).

6. Conclusions

In this study we have applied the technique of Lagrange multipliers during the investigation of the optimization problems. We have taken attempts to discuss utility maximization policy subject to two constraints: budget constraint and coupon constraint. We have tried to provide sensitivity analysis, that is, we have tried to discuss the situation, for example, if the price of a certain commodity rises, how an individual consumer behaves. We have used four commodity variables to develop the paper successfully. When we face difficulties working with four commodity variables, we have made some assumptions, such as we consider two commodities equal to unity. Later, we have considered all commodities are of unit amount, and prices of two commodities are same, and also two types of coupon numbers are same. In the study we have tried to show mathematical calculations in some details.

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