Paradigm Academic Press
Law and Economy
ISSN 2788-7049
JUL. 2023 VOL.2, NO.7

An Economical Study When Cost of Irregular Raw Materials of an Industry Increases for Nonlinear Budget Constraint

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doi:10.56397/LE.2023.07.04

Abstract

Very few researches try to work with nonlinear budget constraint. This study attempts to discuss the sensitivity analysis with the use of nonlinear budget constraint. For the efficient use of inputs every industry should inspects economic analysis for the profit maximization. In this study Lagrange multiplier technique is applied with the bordered Hessian and Jacobian. For the sustainability of an industry in the competitive global economy, the industry should proceed through the application of scientific method, such as application of optimization policy.

Keywords: profit maximization, nonlinear budget constraint, irregular input

1. Introduction

Mathematical modeling finds its floor in field of social sciences, such as in economics, sociology, psychology, political science, etc. (Samuelson, 1947; Carter, 2001). On the other hand, mathematical modeling in economics is the application of mathematics in economics to represent theories and analyze problems (Mohajan, 2018b). It plays a leading role in modern economics to analyze optimization (Ferdous & Mohajan, 2022). In the society economists and sociologists search for the social benefits and their own benefits (Eaton & Lipsey, 1975). At present optimization investigation becomes a fundamental concept in business, economics, industry, factory, firm, and some other related fields (Samuelson, 1947).

In this study we have used Cobb-Douglas production function as our profit function (Cobb & Douglas, 1928). We have developed this study through the analysis of determinant of 6×6 bordered Hessian matrix and 6×6 Jacobian to predict on the future production. We are always sincere to show the mathematical calculations very clearly.

2. Literature Review

A literature review is an overview of the previously published works, which allows a researcher to identify relevant theories, methods, and gaps in the existing research (Creswell, 2007). It is an introductory section of a research, which highlights the contributions of other scholars in the same field within the existing knowledge (Polit & Hungler, 2013). A good literature review can ensure that a proper research question has been asked and a proper research methodology has been chosen (Torraco, 2016). Cobb-Douglas production function is a seminal work of two American scholars; mathematician Charles W. Cobb (1875-1949) and economist Paul H. Douglas (1892-1976) that can be utilized in profit maximization studies (Cobb & Douglas, 1928). Another two US professors; mathematician John V. Baxley and economist John C. Moorhouse have worked on optimization with sufficient mathematical techniques (Baxley & Moorhouse, 1984).

Famous mathematician Jamal Nazrul Islam (1939-2013) and his coauthors have discussed both profit maximization and utility maximization (Islam et al., 2010, 2011). Lia Roy and her coauthors have studied cost minimization, where they have utilized Cobb-Douglas production function (Roy et al., 2021). Pahlaj Moolio and his coworkers have also worked very eagerly on profit maximization (Moolio et al., 2009). Jannatul Ferdous and Haradhan Kumar Mohajan have discussed both necessary and sufficient conditions to determine the profit maximization and to verify it (Ferdous & Mohajan, 2022). Devajit Mohajan and Haradhan Kumar Mohajan have worked on a series of papers to show the optimum results in economics (Mohajan & Mohajan, 2022a-f, 2023a-g).

3. Research Methodology of the Study

Research is a logical and systematic search for new useful information on a specific topic (Rajasekar et. al., 2013). Methodology is the systematic and theoretical analysis of the methods applied to a field of study (Patel & Patel, 2019). Sandra Harding has tried to find a relationship between method and methodology as: method is "*techniques for gathering evidence*"; whereas methodology is "*a theory and analysis of how research does or should proceed*" (Harding, 1987). Methodology makes relationship with the nature and power to science, truth, and epistemology (Ramazanoglu & Holland, 2002). It is the guideline of a research work (Kothari, 2008). It shows the research design and analysis procedures (Hallberg, 2006). Therefore, research methodology is the procedure to perform a research in a systematic way (Abbasi, 2015). To choose a research methodology, a researcher must understand its philosophical origins and unique characteristics (Mohajan, 2017b; Rieger, 2019).

In this study we have depended on the optimization related mathematical secondary data sources (Islam et al., 2009a, b; Mohajan, 2017a, 2018b, 2020). These are collected from research articles of renowned journals, books of famous authors, internet, websites, etc. (Mohajan, 2018a, 2021a, b).

4. Objective of the Study

The principal objective of this article is to analyze the economic effects of various inputs when the cost of irregular raw material is increased. Other minor objectives of the study are as follows:

- to show the mathematical terms elaborately, and
- to provide the economic results precisely.

5. Lagrangian Function

We consider that an industry tries to make a maximum profit from its products. Let the industry uses D_1 amount of capital, D_2 quantity of labor, D_3 quantity of principal raw materials, and D_4 quantity of irregular raw material for its annual production. Let us consider the Cobb-Douglas production function $f(D_1, D_2, D_3, D_4)$ as a profit function for our model (Cobb & Douglas, 1928; Mohajan & Mohajan, 2022a),

$$P(D_1, D_2, D_3, D_4) = f(D_1, D_2, D_3, D_4) = AD_1^{\alpha} D_2^{\beta} D_3^{\gamma} D_4^{\delta}, \quad (1)$$

where *A* is the efficiency parameter that reflects the level of technology, i.e., technical process, economic system, etc., which represents total factor productivity. Moreover, *A* reflects the skill and efficient level of the workforce. Here α, β, γ, and δ are parameters; α indicates the output of elasticity of capital measures the percentage change in $P(D_1, D_2, D_3, D_4)$ for 1% change in D_1, while D_2, D_3, and D_4 are held constants. Similarly, β indicates the output of elasticity of labor, γ indicates the output of elasticity of principal raw materials, and δ indicates the output of elasticity of irregular raw material. Now these four parameters α, β, γ, and δ must satisfy the following four inequalities (Islam et al., 2010; Moolio et al., 2009; Mohajan, 2022):

$$0 < \alpha < 1, \; 0 < \beta < 1, \; 0 < \gamma < 1, \text{ and } 0 < \delta < 1. \quad (2)$$

A strict Cobb-Douglas production function, in which $\Re = \alpha + \beta + \gamma + \delta < 1$ indicates decreasing returns to scale, $\Re = 1$ indicates constant returns to scale, and $\Re > 1$ indicates increasing returns to scale. Now we consider that the profit function is subject to a nonlinear budget constraint as (Moolio et al., 2009; Roy et al., 2021; Mohajan & Mohajan, 2022f, 2023d),

$$B(D_1, D_2, D_3, D_4) = kD_1 + lD_2 + mD_3 + n(D_4)D_4, \quad (3)$$

where *k* is rate of interest or services of capital per unit of capital D_1; *l* is the wage rate per unit of labor D_2; *m* is the cost per unit of principal raw material D_3; and *n* is the cost per unit of irregular raw material D_4. In nonlinear budget equation (3) we consider (Mohajan & Mohajan, 2023a),

$$n(E_4) = n_0 E_4 - n_0 , \tag{4}$$

where n_0 being the discounted price of the irregular input D_4. Therefore, the nonlinear budget constraint (3) takes the form (Mohajan, 2021b; Mohajan & Mohajan, 2023e);

$$B(D_1, D_2, D_3, D_4) = kD_1 + lD_2 + mD_3 + n_0 D_4^2 - n_0 D_4 . \tag{5}$$

We now formulate the maximization problem for the profit function (1) in terms of single Lagrange multiplier λ by defining the Lagrangian function $T(D_1, D_2, D_3, D_4, \lambda)$ as (Ferdous & Mohajan, 2022; Mohajan & Mohajan, 2023c),

$$T(D_1, D_2, D_3, D_4, \lambda) = AD_1^{\alpha} D_2^{\beta} D_3^{\gamma} D_4^{\delta} + \lambda\left\{B(D_1, D_2, D_3, D_4) - kD_1 - lD_2 - mD_3 - n_0 D_4^2 + n_0 D_4\right\}. \tag{6}$$

Relation (6) is a 5-dimensional unconstrained problem that is obtained from (1) and 4-dimensional constrained problem (3), where Lagrange multiplier λ, is considered as a device in our profit maximization model.

6. Analysis on Four Inputs

For maximization, first order differentiation equals to zero; then from (6) we can write (Islam et al., 2011; Mohajan, 2021a; Mohajan & Mohajan, 2022g),

$$T_{\lambda} = B - kD_1 - lD_2 - mD_3 - n_0 D_4^2 + n_0 D_4 = 0 , \tag{7a}$$

$$T_1 = \alpha A D_1^{\alpha-1} D_2^{\beta} D_3^{\gamma} D_4^{\delta} - \lambda k = 0 , \tag{7b}$$

$$T_2 = \beta A D_1^{\alpha} D_2^{\beta-1} D_3^{\gamma} D_4^{\delta} - \lambda l = 0 , \tag{7c}$$

$$T_3 = \gamma A D_1^{\alpha} D_2^{\beta} D_3^{\gamma-1} D_4^{\delta} - \lambda m = 0 , \tag{7d}$$

$$T_4 = \delta A D_1^{\alpha} D_2^{\beta} D_3^{\gamma} D_4^{\delta-1} - \lambda n_0 (2D_4 - 1) = 0 , \tag{7e}$$

where, $\frac{\partial T}{\partial \lambda} = T_{\lambda}$, $\frac{\partial T}{\partial E_1} = T_1$, $\frac{\partial T}{\partial E_2} = T_2$, etc. indicate first-order partial differentiations of multivariate Lagrangian function.

Using equations (2) to (7) we can decide the values of D_1, D_2, D_3, and D_4 as follows (Ferdous & Mohajan 2022; Mohajan & Mohajan, 2022b):

$$D_1 = \frac{\alpha B}{k\,\Re} , \tag{8a}$$

$$D_2 = \frac{\beta B}{l\,\Re} , \tag{8b}$$

$$D_3 = \frac{\gamma B}{m\,\Re} , \tag{8c}$$

$$D_4 = \frac{\delta B}{n\,\Re} . \tag{8d}$$

7. Bordered Hessian Matrix Analysis

Let us consider the determinant of the 5×5 bordered Hessian matrix as (Islam et al. 2010; Mohajan & Mohajan, 2023d),

$$|H| = \begin{vmatrix} 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ -B_1 & T_{11} & T_{12} & T_{13} & T_{14} \\ -B_2 & T_{21} & T_{22} & T_{23} & T_{24} \\ -B_3 & T_{31} & T_{32} & T_{33} & T_{34} \\ -B_4 & T_{41} & T_{42} & T_{43} & T_{44} \end{vmatrix}. \quad (9)$$

Taking first-order partial differentiations of (5) we get,

$$B_1 = k, \quad B_2 = l, \quad B_3 = m, \text{ and } \quad B_4 = 2n_0 D_4 - n_0. \quad (10)$$

Taking second-order and cross-partial derivatives of (6) we get (Roy et al., 2021; Mohajan & Mohajan, 2023c),

$$T_{11} = \alpha(\alpha - 1)AD_1^{\alpha-2} D_2^{\beta} D_3^{\gamma} D_4^{\delta},$$

$$T_{22} = \beta(\beta - 1)AD_1^{\alpha} D_2^{\beta-2} D_3^{\gamma} D_4^{\delta},$$

$$T_{33} = \gamma(\gamma - 1)AD_1^{\alpha} D_2^{\beta} D_3^{\gamma-2} D_4^{\delta},$$

$$T_{44} = \delta(\delta - 1)AD_1^{\alpha} D_2^{\beta} D_3^{\gamma} D_4^{\delta-2},$$

$$T_{12} = T_{21} = \alpha\beta AD_1^{\alpha-1} D_2^{\beta-1} D_3^{\gamma} D_4^{\delta},$$

$$T_{13} = T_{31} = \alpha\gamma AD_1^{\alpha-1} D_2^{\beta} D_3^{\gamma-1} D_4^{\delta},$$

$$T_{14} = T_{41} = \alpha\delta AD_1^{\alpha-1} D_2^{\beta} D_3^{\gamma} D_4^{\delta-1}, \quad (11)$$

$$T_{23} = T_{32} = \beta\gamma AD_1^{\alpha} D_2^{\beta-1} D_3^{\gamma-1} D_4^{\delta},$$

$$T_{24} = T_{42} = \beta\delta AD_1^{\alpha} D_2^{\beta-1} D_3^{\gamma} D_4^{\delta-1},$$

$$T_{34} = T_{43} = \gamma\delta AD_1^{\alpha} D_2^{\beta} D_3^{\gamma-1} D_4^{\delta-1}.$$

where $\frac{\partial^2 T}{\partial D_1 \partial D_2} = T_{12} = T_{21}$, $\frac{\partial^2 T}{\partial D_2^2} = T_{22}$, etc. indicate cross-partial, second order differentiations of multivariate Lagrangian function, respectively, etc.

Now we expand the Hessian (9) as $|H| > 0$ (Moolio et al., 2009; Mohajan et al., 2013; Mohajan & Mohajan, 2023f),

$$|H| = \frac{A^3 \alpha\beta\gamma\delta D_1^{3\alpha} D_2^{3\beta} D_3^{3\gamma} D_4^{3\delta} B^2}{D_1^2 D_2^2 D_3^2 D_4^2 \Re^2} (\alpha + \beta + \gamma + \delta)(\delta + 3) > 0, \quad (12)$$

where efficiency parameter, $A > 0$, and budget of the firm, $B > 0$; D_1, D_2, D_3, and D_4 are four different types of inputs; and consequently, $D_1, D_2, D_3, D_4 > 0$. Parameters, $\alpha, \beta, \gamma, \delta > 0$; also in the model either $0 < \Re = \alpha + \beta + \gamma + \delta < 1$, $\Re = 1$ or $\Re > 1$. Hence, equation (12) gives; $|H| > 0$ (Islam et al., 2010; Mohajan & Mohajan, 2022c, 2023d).

8. Determination of Lagrange Multiplier λ

Now using the necessary values from (8) in (7a) we get (Roy et al., 2021; Mohajan & Mohajan, 2023f),

$$B = \frac{\alpha A D_1^{\alpha} D_1^{\beta} D_1^{\gamma} D_1^{\delta}}{\lambda} + \frac{\beta A D_1^{\alpha} D_1^{\beta} D_1^{\gamma} D_1^{\delta}}{\lambda} + \frac{\gamma A D_1^{\alpha} D_1^{\beta} D_1^{\gamma} D_1^{\delta}}{\lambda} + \frac{\delta A D_1^{\alpha} D_1^{\beta} D_1^{\gamma} D_1^{\delta}}{\lambda}$$

$$\lambda = \frac{A D_1^{\alpha} D_1^{\beta} D_1^{\gamma} D_1^{\delta} \Re}{B}. \qquad (13)$$

9. Jacobian Matrix Analysis

We have observed that the second-order condition is satisfied, so that the determinant of (5) survives at the optimum, i.e., $|J| = |H|$; and hence, we can apply the implicit function theorem. Now we compute twenty-five partial derivatives, such as $\frac{\partial \lambda}{\partial k}$, $\frac{\partial D_1}{\partial k}$, $\frac{\partial D_3}{\partial l}$, $\frac{\partial D_4}{\partial B}$, etc. that are referred to as the comparative statics of the model (Chiang, 1984; Mohajan & Mohajan, 2022a).

Let $\mathbf{G}$ be the vector-valued function of ten variables $\lambda^*, D_1^*, D_2^*, D_3^*, D_4^*, k, l, m, n$, and *B*, and we define the function $\mathbf{G}$ for the point $\left(\lambda^*, D_1^*, D_2^*, D_3^*, D_4^*, k, l, m, n, B\right) \in R^{10}$, and take the values in R^5. By the Implicit Function Theorem of multivariable calculus, the equation (Mohajan, 2021b; Mohajan & Mohajan, 2022f, 2023b),

$$F\left(\lambda^*, D_1^*, D_2^*, D_3^*, D_4^*, k, l, m, n, B\right) = 0, \qquad (14)$$

may be solved in the form of

$$\begin{bmatrix} \lambda \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} = \mathbf{G}(k, l, m, n, B). \qquad (15)$$

Now the 5×5 Jacobian matrix for $\mathbf{G}(k, l, m, n, B)$; regarded as $J_G = \frac{\partial(\lambda, D_1, D_2, D_3, D_4)}{\partial(k, l, m, n_0, B)}$, and is represented by;

$$J_G = \begin{bmatrix} \frac{\partial \lambda}{\partial k} & \frac{\partial \lambda}{\partial l} & \frac{\partial \lambda}{\partial m} & \frac{\partial \lambda}{\partial n_0} & \frac{\partial \lambda}{\partial B} \\ \frac{\partial D_1}{\partial k} & \frac{\partial D_1}{\partial l} & \frac{\partial D_1}{\partial m} & \frac{\partial D_1}{\partial n_0} & \frac{\partial D_1}{\partial B} \\ \frac{\partial D_2}{\partial k} & \frac{\partial D_2}{\partial l} & \frac{\partial D_2}{\partial m} & \frac{\partial D_2}{\partial n_0} & \frac{\partial D_2}{\partial B} \\ \frac{\partial D_3}{\partial k} & \frac{\partial D_3}{\partial l} & \frac{\partial D_3}{\partial m} & \frac{\partial D_3}{\partial n_0} & \frac{\partial D_3}{\partial B} \\ \frac{\partial D_4}{\partial k} & \frac{\partial D_4}{\partial l} & \frac{\partial D_4}{\partial m} & \frac{\partial D_4}{\partial n_0} & \frac{\partial D_4}{\partial B} \end{bmatrix}. \qquad (16)$$

$$= -J^{-1}\begin{bmatrix} -D_1 & -D_2 & -D_3 & -D_4^2+D_4 & 1 \\ -\lambda & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & -2\lambda D_4+\lambda & 0 \end{bmatrix}.$$

The inverse of Jacobian matrix is, $J^{-1} = \frac{1}{|J|}C^T$, where $C = (C_{ij})$, the matrix of cofactors of J, where T for transpose, then (16) becomes (Moolio et al., 2009; Roy et al., 2021; Mohajan, 2021c),

$$= -\frac{1}{|J|}\begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} & C_{51} \\ C_{12} & C_{22} & C_{32} & C_{42} & C_{52} \\ C_{13} & C_{23} & C_{33} & C_{43} & C_{53} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{54} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} \end{bmatrix}\begin{bmatrix} -D_1 & -D_2 & -D_3 & -D_4^2+D_4 & 1 \\ -\lambda & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & -2\lambda D_4+\lambda & 0 \end{bmatrix}$$

$$J_G = -\frac{1}{|J|}\begin{bmatrix} -D_1C_{11}-\lambda C_{21} & -D_2C_{11}-\lambda C_{31} & -D_3C_{11}-\lambda C_{41} & -D_4^2C_{11}+D_4C_{11}-2\lambda D_4C_{51}+\lambda C_{51} & C_{11} \\ -D_1C_{12}-\lambda C_{22} & -D_2C_{12}-\lambda C_{32} & -D_3C_{12}-\lambda C_{42} & -D_4^2C_{12}+D_4C_{12}-2\lambda D_4C_{52}+\lambda C_{52} & C_{12} \\ -D_1C_{13}-\lambda C_{23} & -D_2C_{13}-\lambda C_{33} & -D_3C_{13}-\lambda C_{43} & -D_4^2C_{13}+D_4C_{13}-2\lambda D_4C_{53}+\lambda C_{53} & C_{13} \\ -D_1C_{14}-\lambda C_{24} & -D_2C_{14}-\lambda C_{34} & -D_3C_{14}-\lambda C_{44} & -D_4^2C_{14}+D_4C_{14}-2\lambda D_4C_{54}+\lambda C_{54} & C_{14} \\ -D_1C_{15}-\lambda C_{25} & -D_2C_{15}-\lambda C_{35} & -D_3C_{15}-\lambda C_{45} & -D_4^2C_{15}+D_4C_{15}-2\lambda D_4C_{55}+\lambda C_{55} & C_{15} \end{bmatrix}. \quad (17)$$

In (17) total 25 comparative statics are available, and in this study, we deal only with four of them when discounted price of the irregular input is increased. The firm always attempts for the profit maximization production (Baxley & Moorhouse, 1984; Islam et al., 2010).

10. Sensitivity Analysis

Now we analyze the effect on capital D_1 when the discounted price of the irregular raw material, n_0 increases. Taking T_{24} (i.e., term of 2nd row and 4th column) from both sides of (17) we get (Moolio et al., 2009; Islam et al., 2011; Mohajan, 2018b),

$$\frac{\partial D_1}{\partial n_0} = -\frac{D_4^2}{|J|}[C_{12}] + \frac{D_4}{|J|}[C_{12}] - 2\lambda\frac{D_4}{|J|}[C_{52}] + \frac{\lambda}{|J|}[C_{52}]$$

$$= -\frac{D_4^2}{|J|}\text{Cofactor of } C_{12} + \frac{D_4}{|J|}\text{Cofactor of } C_{12} - 2\lambda\frac{D_4}{|J|}\text{Cofactor of } \mathrm{C}_{52} + \frac{\lambda}{|J|}\text{Cofactor of } \mathrm{C}_{52}$$

$$= -\frac{D_4^2-D_4}{|J|}\begin{vmatrix} -B_1 & T_{12} & T_{13} & T_{14} \\ -B_2 & T_{22} & T_{23} & T_{24} \\ -B_3 & T_{32} & T_{33} & T_{34} \\ -B_4 & T_{42} & T_{43} & T_{44} \end{vmatrix} - \frac{\lambda(1-2D_4)}{|J|}\begin{vmatrix} 0 & -B_2 & -B_3 & -B_4 \\ -B_1 & T_{12} & T_{13} & T_{14} \\ -B_2 & T_{22} & T_{23} & T_{24} \\ -B_3 & T_{32} & T_{33} & T_{34} \end{vmatrix}$$

$$= -\frac{D_4^2-D_4}{|J|}\left\{-B_1\begin{vmatrix} T_{22} & T_{23} & T_{24} \\ T_{32} & T_{33} & T_{34} \\ T_{42} & T_{43} & T_{44} \end{vmatrix} - T_{12}\begin{vmatrix} -B_2 & T_{23} & T_{24} \\ -B_3 & T_{33} & T_{34} \\ -B_4 & T_{43} & T_{44} \end{vmatrix} + T_{13}\begin{vmatrix} -B_2 & T_{22} & T_{24} \\ -B_3 & T_{32} & T_{34} \\ -B_4 & T_{42} & T_{44} \end{vmatrix} - T_{14}\begin{vmatrix} -B_2 & T_{22} & T_{23} \\ -B_3 & T_{32} & T_{33} \\ -B_4 & T_{42} & T_{43} \end{vmatrix}\right\}$$

$$-\frac{\lambda(1-2D_4)}{|J|}\left\{B_2\begin{vmatrix}-B_1 & T_{13} & T_{14}\\ -B_2 & T_{23} & T_{24}\\ -B_3 & T_{33} & T_{34}\end{vmatrix}-B_3\begin{vmatrix}-B_1 & T_{12} & T_{14}\\ -B_2 & T_{22} & T_{24}\\ -B_3 & T_{32} & T_{34}\end{vmatrix}+B_4\begin{vmatrix}-B_1 & T_{12} & T_{13}\\ -B_2 & T_{22} & T_{23}\\ -B_3 & T_{32} & T_{33}\end{vmatrix}\right\}$$

$$=-\frac{D_4^2-D_4}{|J|}\left[-B_1\{T_{22}(T_{33}T_{44}-T_{43}T_{34})+T_{23}(T_{42}T_{34}-T_{32}T_{44})+T_{24}(T_{32}T_{43}-T_{42}T_{33})\}\right.$$

$$-T_{12}\{-B_2(T_{33}T_{44}-T_{43}T_{34})+T_{23}(-B_4T_{34}+B_3T_{44})+T_{24}(-B_3T_{43}+B_4T_{33})\}$$

$$+T_{13}\{-B_2(T_{32}T_{44}-T_{42}T_{34})+T_{22}(-B_4T_{34}+B_3T_{44})+T_{24}(-B_3T_{42}+B_4T_{32})\}$$

$$\left.-T_{14}\{-B_2(T_{32}T_{43}-T_{42}T_{33})+T_{22}(-B_4T_{33}+B_3T_{43})+T_{23}(-B_3T_{42}+B_4T_{32})\}\right]$$

$$-\frac{\lambda(1-2D_4)}{|J|}\left[B_2\{-B_1(T_{23}T_{34}-T_{33}T_{24})+T_{13}(-B_3T_{24}+B_2T_{34})+T_{14}(-B_2T_{33}+B_3T_{23})\}\right.$$

$$-B_3\{-B_1(T_{22}T_{34}-T_{32}T_{24})+T_{12}(-B_3T_{24}+B_2T_{34})+T_{14}(-B_2T_{32}+B_3T_{22})\}$$

$$\left.+B_4\{-B_1(T_{22}T_{33}-T_{32}T_{23})+T_{12}(-B_3T_{23}+B_2T_{33})+T_{13}(-B_2T_{32}+B_3T_{22})\}\right]$$

$$=-\frac{D_4^2-D_4}{|J|}\{-B_1T_{22}T_{33}T_{44}+B_1T_{22}T_{34}^2-B_1T_{23}T_{42}T_{34}+B_1T_{23}^2T_{44}-B_1T_{24}T_{32}T_{43}+B_1T_{24}^2T_{33}+B_2T_{12}T_{33}T_{44}$$

$$-B_2T_{12}T_{34}^2+B_4T_{12}T_{23}T_{34}-B_3T_{12}T_{23}T_{44}+B_3T_{12}T_{24}T_{43}-B_4T_{12}T_{24}T_{33}-B_2T_{13}T_{32}T_{44}+B_2T_{13}T_{42}T_{34}$$

$$-B_4T_{13}T_{22}T_{34}+B_3T_{13}T_{22}T_{44}-B_3T_{13}T_{24}^2+B_4T_{13}T_{24}T_{32}+B_2T_{14}T_{32}T_{43}-B_2T_{14}T_{42}T_{33}+B_4T_{14}T_{22}T_{33}$$

$$-B_3T_{14}T_{22}T_{43}+B_3T_{14}T_{23}T_{42}-B_4T_{14}T_{23}^2\}-\frac{\lambda(1-2D_4)}{|J|}\{-B_1B_2T_{23}T_{34}+B_1B_2T_{33}T_{24}+B_2B_3T_{13}T_{24}$$

$$-B_2^2T_{13}T_{34}+B_2^2T_{14}T_{33}-B_2B_3T_{14}T_{23}+B_1B_3T_{22}T_{34}-B_1B_3T_{32}T_{24}+B_3^2T_{12}T_{24}-B_2B_3T_{12}T_{34}+B_2B_3T_{14}T_{32}$$

$$-B_3^2T_{14}T_{22}-B_1B_4T_{22}T_{33}+B_1B_4T_{32}T_{23}+B_3B_4T_{12}T_{23}+B_2B_4T_{12}T_{33}-B_2B_4T_{13}T_{32}+B_3B_4T_{13}T_{22}\}$$

$$=-\frac{D_4^2-D_4}{|J|}\frac{A^3D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}}{D_1^2D_2^2D_3^2D_4^2}\{-kD_1^2\beta(\beta-1)\gamma(\gamma-1)\delta(\delta-1)+kD_1^2\beta(\beta-1)\gamma^2\delta^2-kD_1^2\beta^2\gamma^2\delta^2$$

$$+kD_1^2\beta^2\gamma^2\delta(\delta-1)-kD_1^2\beta^2\gamma^2\delta^2+kD_1^2\beta^2\gamma(\gamma-1)\delta^2+lD_1D_2\alpha\beta\gamma(\gamma-1)\delta(\delta-1)-lD_1D_2\alpha\beta\gamma^2\delta^2$$

$$+nD_1D_4\alpha\beta^2\gamma^2\delta-mD_1D_3\alpha\beta^2\gamma\delta(\delta-1)+mD_1D_3\alpha\beta^2\gamma\delta^2-nD_1D_4\alpha\beta^2\gamma(\gamma-1)\delta$$

$$-lD_1D_2\alpha\beta\gamma^2\delta(\delta-1)+lD_1D_2\alpha\beta\gamma^2\delta^2-nD_1D_4\alpha\beta(\beta-1)\gamma^2\delta+mD_1D_3\alpha\beta(\beta-1)\gamma\delta(\delta-1)$$

$$-mD_1D_3\alpha\beta^2\gamma\delta^2+nD_1D_4\alpha\beta^2\gamma^2\delta+lD_1D_2\alpha\beta\gamma^2\delta^2-lD_1D_2\alpha\beta\gamma(\gamma-1)\delta^2$$

$$+nD_1D_4\alpha\beta(\beta-1)\gamma(\gamma-1)\delta \quad -mD_1D_3\alpha\beta(\beta-1)\gamma\delta^2 \quad +mD_1D_3\alpha\beta^2\gamma\delta^2 \quad -nD_1D_4\alpha\beta^2\gamma^2\delta\}$$

$$-\frac{\lambda(1-2D_4)}{|J|}\frac{A^2D_1^{2x}D_2^{2y}D_3^{2z}D_4^{2w}}{D_1^2D_2^2D_3^2D_4^2}\{-klD_1^2D_2D_4\beta\gamma^2\delta \; +klD_1^2D_2D_4\beta\gamma(\gamma-1)\delta \; +lmD_1D_2D_3D_4\alpha\beta\gamma\delta$$

$$-l^2D_1D_2^2D_4\alpha\gamma^2\delta \quad +l^2D_1D_2^2D_4\alpha(\gamma-1)\delta \quad -lmD_1D_2D_3D_4\alpha\beta\gamma\delta \quad +kmD_1^2D_3D_4\beta(\beta-1)\gamma\delta$$

$$-kmD_1^2D_3D_4\beta^2\gamma\delta \quad +m^2D_1D_3^2D_4\alpha\beta^2\delta \quad -lmD_1D_2D_3D_4\alpha\beta\gamma\delta \quad +lmD_1D_2D_3D_4\alpha\beta\gamma\delta$$

$$-m^2D_1D_3^2D_4\alpha\beta(\beta-1)\delta \quad -knD_1^2D_4^2\beta(\beta-1)\gamma(\gamma-1) \quad +knD_1^2D_4^2\beta^2\gamma^2 \quad +mnD_1D_3D_4^2\alpha\beta^2\gamma$$

$$+nlD_1D_2D_4^2\alpha\beta\gamma(\gamma-1)-nlD_1D_2D_4^2\alpha\beta\gamma^2+mnD_1D_3D_4^2\alpha\beta(\beta-1)\gamma\}$$

$$=-\frac{D_4^2-D_4}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}}{D_1^2D_2^2D_3^2D_4^2}\quad\{-kD_1^2\alpha^{-1}(\beta-1)(\gamma-1)(\delta-1) \quad +kD_1^2\alpha^{-1}(\beta-1)\gamma\delta$$

$$-kD_1^2\alpha^{-1}\beta\gamma\delta \quad +kD_1^2\alpha^{-1}\beta\gamma(\delta-1) \quad -kD_1^2\alpha^{-1}\beta\gamma\delta \quad +kD_1^2\alpha^{-1}\beta(\gamma-1)\delta \quad +lD_1D_2(\gamma-1)(\delta-1)$$

$$-lD_1D_2\gamma\delta \quad +nD_1D_4\beta\gamma \quad -mD_1D_3\beta(\delta-1) \quad +mD_1D_3\beta\delta \quad -nD_1D_4\beta(\gamma-1) \quad -lD_1D_2\gamma(\delta-1)$$

$$+lD_1D_2\gamma\delta \quad -nD_1D_4(\beta-1)\gamma \quad +mD_1D_3(\beta-1)(\delta-1) \quad -mD_1D_3\beta\delta \quad +nD_1D_4\beta\gamma \quad +lD_1D_2\gamma\delta$$

$$-lD_1D_2(\gamma-1)\delta \quad +nD_1D_4(\beta-1)(\gamma-1) \quad -mD_1D_3(\beta-1)\delta \quad +mD_1D_3\beta\delta \quad -nD_1D_4\beta\gamma\}$$

$$-\frac{(1-2D_4)}{|J|}\frac{A^2D_1^{2\alpha}D_2^{2\beta}D_3^{2\gamma}D_4^{2\delta}}{D_1^2D_2^2D_3^2D_4^2}\;\frac{AD_1^{\alpha}D_2^{\beta}D_3^{\gamma}D_4^{\delta}\Re}{B}$$

$$\{-klD_1D_2\beta\gamma^2\delta \quad +klD_1D_2\beta\gamma(\gamma-1)\delta \quad -l^2D_2^2\alpha\gamma^2\delta \quad +l^2D_2^2\alpha\gamma(\gamma-1)\delta \quad +kmD_1D_3\beta(\beta-1)\gamma\delta$$

$$-kmD_1D_3\beta^2\gamma\delta \quad +m^2D_3^2\alpha\beta^2\delta \quad -m^2D_3^2\alpha\beta(\beta-1)\delta \quad -knD_1D_4\alpha(\alpha-1)\gamma(\gamma-1) \quad +knD_1D_4\alpha^2\gamma^2$$

$$+mnD_3D_4\alpha\beta^2\gamma \; +mnD_3D_4\alpha\beta(\beta-1)\gamma \; +nlD_2D_4\alpha\beta\gamma(\gamma-1)-nlD_2D_4\alpha\beta\gamma^2\}$$

$$=-\frac{D_4^2-D_4}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}}{D_1D_2^2D_3^2D_4^2}\quad\{-kD_1\alpha^{-1}(\beta-1)(\gamma-1)(\delta-1) \quad +kD_1\alpha^{-1}(\beta-1)\gamma\delta$$

$$-2kD_1\alpha^{-1}\beta\gamma\delta \quad +kD_1\alpha^{-1}\beta\gamma(\delta-1) \quad +kD_1\alpha^{-1}\beta\gamma(\gamma-1)\delta \quad +lD_2(\gamma-1)(\delta-1) \quad -lD_2\gamma(\delta-1)$$

$$-lD_2(\gamma-1)\delta \quad +lD_2\gamma\delta \quad -mD_3\beta(\delta-1) \quad +mD_3\beta\delta \quad +mD_3(\beta-1)(\delta-1) \quad -mD_3(\beta-1)\delta$$

$$+nD_4(\beta-1)(\gamma-1) \; -nD_4\beta(\gamma-1) \; -nD_4(\beta-1)\gamma \; +nD_4\beta\gamma\} \; -\frac{(1-2D_4)}{|J|}\frac{A^3\beta\gamma D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}\Re}{D_1D_2^2D_3^2D_4B}$$

$$\{-klD_1D_2\beta\gamma^2\delta \quad +klD_1D_2\beta\gamma(\gamma-1)\delta w \quad -l^2D_2^2\alpha\gamma^2\delta \quad +l^2D_2^2\alpha\gamma(\gamma-1)\delta \quad +kmD_1D_3\beta(\beta-1)\gamma\delta$$

$$-kmD_1D_3\beta^2\gamma\delta \quad + m^2D_3^2\alpha\beta^2\delta \quad - m^2D_3^2\alpha\beta(\beta-1)\delta \quad - knD_1D_4\beta(\beta-1)\gamma(\gamma-1) \quad + knD_1D_4\beta^2\gamma^2$$

$$+ mnD_3D_4\alpha\beta^2\gamma + mnD_3D_4\alpha\beta(\beta-1)\gamma + nlD_2D_4\alpha\beta\gamma(\gamma-1) - nlD_2D_4\alpha\beta\gamma^2 \}$$

$$= -\frac{D_4-1}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}B}{D_1D_2^2D_3^2D_4\Re} \quad \{-(\beta-1)(\gamma-1)(\delta-1) \quad + \beta(\gamma-1)(\delta-1) \quad - \beta\gamma(\delta-1)$$

$$+(\beta-1)\gamma(\delta-1) \quad +(2D_4-1)(\beta-1)(\gamma-1)\delta \quad -(2D_4-1)\beta(\gamma-1)\delta \quad -(2D_4-1)(\beta-1)\gamma\delta$$

$$+(2D_4-1)\beta\gamma\delta\} \quad -\frac{(1-2D_4)}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}B}{D_1D_2^2D_3^2D_4\Re} \quad \{-\beta\gamma \quad + \beta(\gamma-1) \quad -(\beta-1)(\gamma-1)$$

$$+(\beta-1)\gamma\}$$

$$= -\frac{D_4-1}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}B}{D_1D_2^2D_3^2D_4\Re}\left(2D_4\delta-2\delta+1\right) - \frac{(1-2D_4)}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}B}{D_1D_2^2D_3^2D_4\Re}$$

$$\frac{\partial D_1}{\partial n_0} = -\frac{1}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}B}{D_1D_2^2D_3^2\Re}\left(2D_4\delta-4\delta-1\right). \tag{18}$$

Now using $\alpha=\beta=\gamma=\delta=\frac{1}{4}$ then we get, $\Re=1$, i.e., for constant returns to scale, in (18) we get,

$$\frac{\partial D_1}{\partial n_0} = \frac{1}{|J|}\frac{A^3D_4^{\frac{3}{4}}B}{2^9D_1^{\frac{1}{4}}D_2^{\frac{5}{4}}D_3^{\frac{5}{4}}}\left(4-D_4\right). \tag{19}$$

Now we consider $D_4 > 4$ in (19) then we see that,

$$\frac{\partial D_1}{\partial n_0} > 0. \tag{20}$$

Inequality (20) indicates that if the discounted price of the irregular raw material, n_0 increases; the amount of capital also increases. It seems that the demand of the commodities and profit of the industry has increased, and the industry has increased its capital structure.

From (19) if $D_4 < 4$ we see that,

$$\frac{\partial D_1}{\partial n_0} < 0. \tag{21}$$

Inequality (21) shows that if the discounted price of the irregular raw material, n_0 increases; the amount of capital decreases, which is reasonable. In this situation the industry may decrease the production due to the shortage of capital. Hence, the industry faces unsustainable circumstances.

Now we consider $D_4 = 4$ in (19) then we see that,

$$\frac{\partial D_1}{\partial n_0} = 0. \tag{22}$$

Equation (22) shows that if the discounted price of the irregular raw material, n_0 increases; there is no change of amount of capital. In this situation it seems that there is no relation between irregular raw material and capital of the industry.

Now we analyze the effect on wage D_2 when the discounted price of the irregular raw material, n_0 increases. Taking T_{34} (i.e., term of 3rd row and 4th column) from both sides of (17) we get (Roy et al., 2021; Mohajan & Mohajan, 2023c),

$$\frac{\partial D_2}{\partial n_0} = -\frac{D_4^2}{|J|}[C_{13}] + \frac{D_4}{|J|}[C_{13}] - 2\lambda\frac{D_4}{|J|}[C_{53}] + \frac{\lambda}{|J|}[C_{53}]$$

$$= -\frac{D_4^2}{|J|}\text{Cofactor of } C_{13} + \frac{D_4}{|J|}\text{Cofactor of } C_{13} - 2\lambda\frac{D_4}{|J|}\text{Cofactor of } \mathrm{C}_{53} + \frac{\lambda}{|J|}\text{Cofactor of } \mathrm{C}_{53}$$

$$= \frac{D_4^2 - D_4}{|J|}\begin{vmatrix} -B_1 & T_{11} & T_{13} & T_{14} \\ -B_2 & T_{21} & T_{23} & T_{24} \\ -B_3 & T_{31} & T_{33} & T_{34} \\ -B_4 & T_{41} & T_{43} & T_{44} \end{vmatrix} + \frac{\lambda(1-2D_4)}{|J|}\begin{vmatrix} 0 & -B_1 & -B_3 & -B_4 \\ -B_1 & T_{11} & T_{13} & T_{14} \\ -B_2 & T_{21} & T_{23} & T_{24} \\ -B_3 & T_{31} & T_{33} & T_{34} \end{vmatrix}$$

$$= \frac{D_4^2 - D_4}{|J|}\left\{ -B_1\begin{vmatrix} T_{21} & T_{23} & T_{24} \\ T_{31} & T_{33} & T_{34} \\ T_{41} & T_{43} & T_{44} \end{vmatrix} - T_{11}\begin{vmatrix} -B_2 & T_{23} & T_{24} \\ -B_3 & T_{33} & T_{34} \\ -B_4 & T_{43} & T_{44} \end{vmatrix} + T_{13}\begin{vmatrix} -B_2 & T_{21} & T_{24} \\ -B_3 & T_{31} & T_{34} \\ -B_4 & T_{41} & T_{44} \end{vmatrix} - T_{14}\begin{vmatrix} -B_2 & T_{21} & T_{23} \\ -B_3 & T_{31} & T_{33} \\ -B_4 & T_{41} & T_{43} \end{vmatrix} \right\}$$

$$+ \frac{\lambda(1-2D_4)}{|J|}\left\{ B_1\begin{vmatrix} -B_1 & T_{13} & T_{14} \\ -B_2 & T_{23} & T_{24} \\ -B_3 & T_{33} & T_{34} \end{vmatrix} - B_3\begin{vmatrix} -B_1 & T_{11} & T_{14} \\ -B_2 & T_{21} & T_{24} \\ -B_3 & T_{31} & T_{34} \end{vmatrix} + B_4\begin{vmatrix} -B_1 & T_{11} & T_{13} \\ -B_2 & T_{21} & T_{23} \\ -B_3 & T_{31} & T_{33} \end{vmatrix} \right\}$$

$$= \frac{D_4^2 - D_4}{|J|}\left[-B_1\{T_{21}(T_{33}T_{44} - T_{43}T_{34}) + T_{23}(T_{41}T_{34} - T_{31}T_{44}) + T_{24}(T_{31}T_{43} - T_{41}T_{33})\} \right.$$

$$-T_{11}\{-B_2(T_{33}T_{44} - T_{43}T_{34}) + T_{23}(-B_4T_{34} + B_3T_{44}) + T_{24}(-B_3T_{43} + B_4T_{33})\}$$

$$+T_{13}\{-B_2(T_{31}T_{44} - T_{41}T_{34}) + T_{21}(-B_4T_{34} + B_3T_{44}) + T_{24}(-B_3T_{41} + B_4T_{31})\}$$

$$\left. -T_{14}\{-B_2(T_{31}T_{43} - T_{41}T_{33}) + T_{21}(-B_4T_{33} + B_3T_{43}) + T_{23}(-B_3T_{41} + B_4T_{31})\}\right]$$

$$+ \frac{\lambda(1-2D_4)}{|J|}\left[B_1\{-B_1(T_{23}T_{34} - T_{33}T_{24}) + T_{13}(-B_3T_{24} + B_2T_{34}) + T_{14}(-B_2T_{33} + B_3T_{23})\} \right.$$

$$-B_3\{-B_1(T_{21}T_{34} - T_{31}T_{24}) + T_{11}(-B_3T_{24} + B_2T_{34}) + T_{14}(-B_2T_{31} + B_3T_{21})\}$$

$$\left. +B_4\{-B_1(T_{21}T_{33} - T_{31}T_{23}) + T_{11}(-B_3T_{23} + B_2T_{33}) + T_{13}(-B_2T_{31} + B_3T_{21})\}\right]$$

$$= -\frac{D_4^2 - D_4}{|J|}\ \{B_1T_{21}T_{33}T_{44} \quad - B_1T_{21}T_{43}T_{34} \quad + B_1T_{23}T_{41}T_{24} \quad - B_1T_{23}T_{31}T_{44} \quad + B_1T_{24}T_{31}T_{43} \quad - B_1T_{24}T_{41}T_{33}$$

$$- B_2T_{11}T_{33}T_{44} \quad + B_2T_{11}T_{43}T_{34} \quad - B_4T_{11}T_{23}T_{34} \quad + B_3T_{11}T_{23}T_{44} \quad - B_3T_{11}T_{24}T_{43} \quad + B_4T_{11}T_{24}T_{33} \quad + B_2T_{13}T_{31}T_{44}$$

$$-B_2T_{13}T_{41}T_{34} + B_4T_{13}T_{21}T_{34} - B_3T_{13}T_{21}T_{44} + B_3T_{13}T_{24}T_{41} - B_4T_{13}T_{24}T_{31} - B_2T_{14}T_{31}T_{43} + B_2T_{14}T_{41}T_{33}$$

$$-B_4T_{14}T_{21}T_{33} + B_3T_{14}T_{21}T_{43} - B_3T_{14}T_{23}T_{41} + B_4T_{14}T_{23}T_{31}\} + \frac{\lambda(1-2D_4)}{|J|}\{B_1^2T_{23}T_{34} - B_1^2T_{33}T_{24}$$

$$+B_1B_3T_{13}T_{24} - B_1B_2T_{13}T_{34} + B_1B_2T_{14}T_{33} - B_1B_3T_{14}T_{23} + B_1B_3T_{21}T_{34} - B_1B_3T_{31}T_{24} + B_3^2T_{11}T_{24}$$

$$-B_2B_3T_{11}T_{34} + B_2B_3T_{14}T_{31} - B_3^2T_{14}T_{21} - B_1B_4T_{21}T_{33} + B_1B_4T_{31}T_{23} - B_3B_4T_{11}T_{23} + B_2B_4T_{11}T_{33}$$

$$-B_2B_4T_{13}T_{31} + B_3B_4T_{13}T_{21}\}$$

$$= -\frac{D_4^2 - D_4}{|J|}\frac{A^3D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}}{D_1^2D_2^2D_3^2D_4^2}\{kD_1D_2\alpha\beta\gamma(\gamma-1)\delta(\delta-1) - kD_1D_2\alpha\beta\gamma^2\delta^2 + kD_1D_2\alpha\beta\gamma^2\delta^2$$

$$-kD_1D_2\alpha\beta\gamma^2\delta(\delta-1) + kD_1D_2\alpha\beta\gamma^2\delta^2 - kD_1D_2\alpha\beta\gamma(\gamma-1)\delta^2 - lD_2^2\alpha(\alpha-1)\gamma(\gamma-1)\delta(\delta-1)$$

$$+lD_2^2\alpha(\alpha-1)\gamma^2\delta^2 - nD_2D_4\alpha(\alpha-1)\beta\gamma^2\delta + mD_2D_3\alpha(\alpha-1)\beta\gamma\delta(\delta-1) - mD_2D_3\alpha(\alpha-1)\beta\gamma\delta^2$$

$$+nD_2D_4\alpha(\alpha-1)\beta\gamma(\gamma-1)\delta + lD_2^2\alpha^2\gamma^2\delta(\delta-1) - lD_2^2\alpha^2\gamma^2\delta^2 + nD_2D_4\alpha^2\beta\gamma^2\delta$$

$$-mD_2D_3\alpha^2\beta\gamma\delta(\delta-1) + mD_2D_3\alpha^2\beta\gamma\delta^2 - nD_2D_4\alpha^2\beta\gamma^2\delta - lD_2^2\alpha^2\gamma^2\delta^2 + lD_2^2\alpha^2\gamma(\gamma-1)\delta^2$$

$$-nD_2D_4\alpha^2\beta\gamma(\gamma-1)\delta + mD_2D_3\alpha^2\beta\gamma\delta^2 - mD_2D_3\alpha^2\beta\gamma\delta^2 + nD_2D_4\alpha^2\beta\gamma^2\delta\}$$

$$+\frac{\lambda(1-2D_4)}{|J|}\frac{A^2D_1^{2\alpha}D_2^{2\beta}D_3^{2\gamma}D_4^{2\delta}}{D_1^2D_2^2D_3^2D_4^2}\{k^2D_1^2D_2D_4\beta\gamma^2\delta - k^2D_1^2D_2D_4\beta\gamma(\gamma-1)\delta + kmD_1D_2D_3D_4\alpha\beta\gamma\delta$$

$$-klD_1D_2^2D_4\alpha\gamma^2\delta + klD_1D_2^2D_4\alpha\gamma(\gamma-1)\delta - kmD_1D_2D_3D_4\alpha\beta\gamma\delta + kmD_1D_2D_3D_4\alpha\beta\gamma\delta$$

$$-kmD_1D_2D_3D_4\alpha\beta\gamma\delta + m^2D_2D_3^2D_4\alpha(\alpha-1)\beta\delta - lmD_2^2D_3D_4\alpha(\alpha-1)\gamma\delta + lmD_2^2D_3D_4\alpha^2\gamma\delta$$

$$-m^2D_2D_3^2D_4\alpha^2\beta\delta - knD_1D_2D_4^2\alpha\beta\delta(\delta-1) + knD_1D_2D_4^2\alpha\beta\delta^2 - mnD_2D_3D_4^2\alpha(\alpha-1)\beta\gamma$$

$$+nlD_2^2D_4^2\alpha(\alpha-1)\gamma(\gamma-1) - nlD_2^2D_4^2\alpha^2\gamma^2 + mnD_2D_3D_4^2\alpha^2\beta\gamma\}$$

$$= -\frac{D_4-1}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}}{D_1^2D_2D_3^2D_4}\{\alpha(\gamma-1)(\delta-1) - (\alpha-1)(\gamma-1)(\delta-1) + (\alpha-1)\gamma(\delta-1)$$

$$-\alpha\gamma(\delta-1) - (2D_4-1)(\alpha-1)\gamma\delta + (2D_4-1)(\alpha-1)(\gamma-1)\delta - (2D_4-1)\alpha(\gamma-1)\delta + (2D_4-1)\alpha\gamma\delta\}$$

$$+\frac{(1-2D_4)}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}}{D_1^2D_2D_3^2D_4}\{(\alpha-1)(\gamma-1)-(\alpha-1)\gamma\}$$

$$= -\frac{D_4-1}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}}{D_1^2D_2D_3^2D_4}(2\delta D_4 - 2\delta + 1) - \frac{(1-2D_4)}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}}{D_1^2D_2D_3^2D_4}(\alpha-1)$$

$$\frac{\partial D_2}{\partial n_0} = -\frac{1}{|J|} \frac{A^3 \alpha\beta\gamma\delta D_1^{3\alpha} D_2^{3\beta} D_3^{3\gamma} D_4^{3\delta}}{D_1^2 D_2 D_3^2 D_4} \left\{2D_4^2\delta - D_4(4\delta + 2\alpha - 3) + \alpha + 2\delta - 2\right\}. \quad (23)$$

Now using $\alpha = \beta = \gamma = \delta = \frac{1}{4}$ then we get, $\Re = 1$, i.e., for constant returns to scale, in (23) we get,

$$\frac{\partial D_2}{\partial n_0} = -\frac{1}{|J|} \frac{A^3 B}{2^{10} D_1^{\frac{5}{4}} D_2^{\frac{1}{4}} D_3^{\frac{5}{4}} D_4^{\frac{1}{4}}} \left(2D_4^2 + 6D_4 - 5\right)$$

$$\frac{\partial D_2}{\partial n_0} = -\frac{1}{|J|} \frac{A^3 B}{2^9 D_1^{\frac{5}{4}} D_2^{\frac{1}{4}} D_3^{\frac{5}{4}} D_4^{\frac{1}{4}}} \left\{\left(D_4 + \frac{3}{2}\right)^2 - \frac{19}{4}\right\}. \quad (24)$$

In (24) if $D_4 > \left(\sqrt{19} - 3\right)/2$ then we get,

$$\frac{\partial D_2}{\partial n_0} < 0. \quad (25)$$

Inequality (25) shows that if the discounted price of the irregular raw material, n_0 increases; the level of workers decrease. In this situation the industry may decrease the production due to the shortage of workers. Hence, the industry faces unsustainable circumstances for constant returns to scale, it can follow the increasing or decreasing returns to scale, but increasing returns to scale will not be favorable if the industry faces shortage of workers.

In (24) if $D_4 < \left(\sqrt{19} - 3\right)/2$ then we get,

$$\frac{\partial D_2}{\partial n_0} > 0. \quad (26)$$

Inequality (26) shows that if the discounted price of the irregular raw material, n_0 increases; the level of workers increase. We have observed that a constant return to scale is suitable for the industry.

In (24) if $D_4 = \left(\sqrt{19} - 3\right)/2$ then we get,

$$\frac{\partial D_2}{\partial n_0} = 0. \quad (27)$$

Equation (27) shows that if the discounted price of the irregular raw material, n_0 increases; there is no change of the level of workers. In this situation it seems that there is no relation between irregular raw material and workers of the industry.

Now using $\alpha = \beta = \gamma = \delta = \frac{1}{2}$ then we get, $\Re = 2$, i.e., for increasing returns to scale, in (23) we get,

$$\frac{\partial D_2}{\partial n_0} = -\frac{1}{|J|} \frac{A^3 B}{2^6 D_1^{\frac{5}{4}} D_3^{\frac{1}{4}} D_3^{\frac{5}{4}} D_3^{\frac{1}{4}}} \left(D_4^2 - \frac{1}{2}\right). \quad (28)$$

In (28) if $D_4 > \frac{1}{\sqrt{2}}$ we get,

$$\frac{\partial D_2}{\partial n_0} < 0. \quad (29)$$

Inequality (29) shows that if the discounted price of the irregular raw material, n_0 increases; the level of workers decrease. Hence, in this situation the industry faces unsustainable circumstances for increasing returns to scale. In this situation constant or decreasing returns to scale may give more benefits to the industry for the profit maximization environment.

In (28) if $D_4 < \frac{1}{\sqrt{2}}$ we get,

$$\frac{\partial D_2}{\partial n_0} > 0. \tag{30}$$

Inequality (30) shows that if the discounted price of the irregular raw material, n_0 increases; the level of workers increase. In this situation it seems that irregular raw material is essential for the industry, and workers faces comfortable environment utilizing this material. As a result, both production and profit of the industry may increase.

In (28) if $D_4 = \frac{1}{\sqrt{2}}$ we get,

$$\frac{\partial D_2}{\partial n_0} = 0. \tag{31}$$

Equation (31) shows that if the discounted price of the irregular raw material, n_0 increases; there is no change of the level of workers. In this situation it seems that there is no relation between irregular raw material and workers of the industry.

Now we analyze the economic effects on principal raw material D_3 when the discounted price of the irregular raw material, n_0 increases. Taking T_{44} (i.e., term of 4th row and 4th column) from both sides of (17) we get (Islam et al., 2010; Mohajan & Mohajan, 2022a, 2023d),

$$\frac{\partial D_3}{\partial n_0} = -\frac{D_4^2}{|J|}[C_{14}] + \frac{D_4}{|J|}[C_{14}] - 2\lambda\frac{D_4}{|J|}[C_{54}] + \frac{\lambda}{|J|}[C_{54}]$$

$$= -\frac{D_4^2}{|J|}\text{Cofactor of } C_{14} + \frac{D_4}{|J|}\text{Cofactor of } C_{14} - 2\lambda\frac{D_4}{|J|}\text{Cofactor of } \mathrm{C}_{54} + \frac{\lambda}{|J|}\text{Cofactor of } \mathrm{C}_{54}$$

$$= -\frac{D_4^2 - D_4}{|J|}\begin{vmatrix} -B_1 & T_{11} & T_{12} & T_{14} \\ -B_2 & T_{21} & T_{22} & T_{24} \\ -B_3 & T_{31} & T_{32} & T_{34} \\ -B_4 & T_{41} & T_{42} & T_{44} \end{vmatrix} - \frac{\lambda(1-2D_4)}{|J|}\begin{vmatrix} 0 & -B_1 & -B_2 & -B_4 \\ -B_1 & T_{11} & T_{12} & T_{14} \\ -B_2 & T_{21} & T_{22} & T_{24} \\ -B_3 & T_{31} & T_{32} & T_{34} \end{vmatrix}$$

$$= -\frac{D_4^2 - D_4}{|J|}\left\{-B_1\begin{vmatrix} T_{21} & T_{22} & T_{24} \\ T_{31} & T_{32} & T_{34} \\ T_{41} & T_{42} & T_{44} \end{vmatrix} - T_{11}\begin{vmatrix} -B_2 & T_{22} & T_{24} \\ -B_3 & T_{32} & T_{34} \\ -B_4 & T_{42} & T_{44} \end{vmatrix} + T_{12}\begin{vmatrix} -B_2 & T_{21} & T_{24} \\ -B_3 & T_{31} & T_{34} \\ -B_4 & T_{41} & T_{44} \end{vmatrix} - T_{14}\begin{vmatrix} -B_2 & T_{21} & T_{22} \\ -B_3 & T_{31} & T_{32} \\ -B_4 & T_{41} & T_{42} \end{vmatrix}\right\}$$

$$-\frac{\lambda(1-2D_4)}{|J|}\left\{B_1\begin{vmatrix} -B_1 & T_{12} & T_{14} \\ -B_2 & T_{22} & T_{24} \\ -B_3 & T_{32} & T_{34} \end{vmatrix} - B_2\begin{vmatrix} -B_1 & T_{11} & T_{14} \\ -B_2 & T_{21} & T_{24} \\ -B_3 & T_{31} & T_{34} \end{vmatrix} + B_4\begin{vmatrix} -B_1 & T_{11} & T_{12} \\ -B_2 & T_{21} & T_{22} \\ -B_3 & T_{31} & T_{32} \end{vmatrix}\right\}$$

$$= -\frac{D_4^2 - D_4}{|J|}\left[-B_1\{T_{21}(T_{32}T_{44} - T_{42}T_{34}) + T_{22}(T_{41}T_{34} - T_{31}T_{44}) + T_{24}(T_{31}T_{42} - T_{41}T_{32})\}\right.$$

$$-T_{11}\{-B_2(T_{32}T_{44} - T_{42}T_{34}) + T_{22}(-B_4T_{34} + B_3T_{44}) + T_{24}(-B_3T_{42} + B_4T_{32})\}$$

$$+T_{12}\{-B_2(T_{31}T_{44} - T_{41}T_{34}) + T_{21}(-B_4T_{34} + B_3T_{44}) + T_{24}(-B_3T_{41} + B_4U_{31})\}$$

$$\left.-T_{14}\{-B_2(T_{31}T_{42} - T_{41}T_{32}) + T_{21}(-B_4T_{32} + B_3T_{42}) + T_{22}(-B_3T_{41} + B_4T_{31})\}\right]$$

$$-\frac{\lambda(1-2D_4)}{|J|}\left[B_1\{-B_1(T_{22}T_{34} - T_{32}T_{24}) + T_{12}(-B_3T_{24} + B_2T_{34}) + T_{14}(-B_2T_{32} + B_3T_{22})\}\right.$$

$$-B_2\{-B_1(T_{21}T_{34} - T_{31}T_{24}) + T_{11}(-B_3T_{24} + B_2T_{34}) + T_{14}(-B_2T_{31} + B_3T_{21})\}$$

$$\left.+B_4\{-B_1(T_{21}T_{32} - T_{31}T_{22}) + T_{11}(-B_3T_{22} + B_2T_{32}) + T_{12}(-B_2T_{31} + B_3T_{21})\}\right]$$

$$= -\frac{D_4^2 - D_4}{|J|}\{-B_1T_{21}T_{32}T_{44} \quad + B_1T_{21}T_{42}T_{34} \quad - B_1T_{22}T_{41}T_{34} \quad + B_1T_{22}T_{31}T_{44} \quad - B_1T_{24}T_{31}T_{42}$$

$$+B_1T_{24}T_{41}T_{32}$$

$$+B_2T_{11}T_{32}T_{44} \quad - B_2T_{11}T_{42}T_{34} \quad + B_4T_{11}T_{22}T_{34} \quad - B_3T_{11}T_{22}T_{44} \quad + B_3T_{11}T_{24}T_{42} - B_4T_{11}T_{24}T_{32} \quad - B_2T_{12}T_{31}T_{44}$$

$$+B_2T_{12}T_{41}T_{34} \quad - B_4T_{12}T_{21}T_{34} \quad + B_3T_{12}T_{21}T_{44} \quad - B_3T_{12}T_{24}T_{41} + B_4T_{12}T_{24}T_{31} \quad + B_2T_{14}T_{31}T_{42} \quad - B_2T_{14}T_{41}T_{32}$$

$$+B_4T_{14}T_{21}T_{32} \quad - B_3T_{14}T_{21}T_{42} \quad + B_3T_{14}T_{22}T_{41} \quad - B_4T_{14}T_{22}T_{31}\} \; - \frac{\lambda(1-2D_4)}{|J|}\{-B_1^2T_{22}T_{34} \quad + B_1^2T_{32}T_{24}$$

$$-B_1B_3T_{12}T_{24} \quad + B_1B_2T_{12}T_{34} \quad - B_1B_2T_{14}T_{32} \quad + B_1B_3T_{14}T_{22} \quad + B_1B_2T_{21}T_{34} \quad - B_1B_2T_{31}T_{24} \quad + B_2B_3T_{11}T_{24}$$

$$-B_2^2T_{11}T_{34} \quad + B_2^2T_{14}T_{31} \quad - B_2B_3T_{14}T_{21} \quad - B_1B_4T_{21}T_{32} \quad + B_1B_4T_{31}T_{22} \quad - B_3B_4T_{11}T_{22} \quad + B_2B_4T_{11}T_{32}$$

$$-B_2B_4T_{12}T_{31} + B_3B_4T_{12}T_{21}\}$$

$$= -\frac{D_4^2 - D_4}{|J|}\frac{A^3D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}}{D_1^2D_2^2D_3^2D_4^2} \quad \{-kD_1D_3\alpha\beta^2\gamma\delta(\delta-1) \quad + kD_1D_3\alpha\beta^2\gamma\delta^2 \quad - kD_1D_3\alpha\beta(\beta-1)\gamma\delta^2$$

$$+kD_1D_3\alpha\beta(\beta-1)\gamma\delta(\delta-1) \quad - kD_1D_3\alpha\beta^2\gamma\delta^2 \quad + kD_1D_3\alpha\beta^2\gamma\delta^2 \quad + lD_2D_3\alpha(\beta-1)\beta\gamma\delta(\delta-1)$$

$$-lD_2D_3\alpha(\alpha-1)\beta\gamma\delta^2 \quad + nD_3D_4\alpha(\alpha-1)\beta(\beta-1)\gamma\delta \quad - mD_3^2\alpha(\alpha-1)\beta(\beta-1)\delta(\delta-1)$$

$$+mD_3^2\alpha(\alpha-1)\beta^2\delta^2 \quad - nD_3D_4\alpha(\alpha-1)\beta^2\gamma\delta \quad - lD_2D_3\alpha^2\beta\gamma\delta(\delta-1) \quad + lD_2D_3\alpha^2\beta\gamma\delta^2$$

$$-nD_3D_4\alpha^2\beta^2\gamma\delta \quad + mD_3^2\alpha^2\beta^2\delta(\delta-1) \quad - mD_3^2\alpha^2\beta^2\delta^2 \quad + nD_3D_4\alpha^2\beta^2\gamma\delta \quad + lD_2D_3\alpha^2\beta\gamma\delta^2$$

$$-lD_2D_3\alpha^2\beta\gamma\delta^2 \ +nD_3D_4\alpha^2\beta^2\gamma\delta \ -mD_3^2\alpha^2\beta^2\delta^2 \ +mD_3^2\alpha^2\beta(\beta-1)\delta^2 \ -nD_3D_4\alpha^2\beta(\beta-1)\gamma\delta\Big\}$$

$$+\frac{\lambda(1-2D_4)}{|J|}\frac{A^2D_1^{2\alpha}D_2^{2\beta}D_3^{2\gamma}D_4^{2\delta}}{D_1^2D_2^2D_3^2D_4^2}\Big\{-k^2D_1^2D_3D_4\beta(\beta-1)\gamma\delta \ +k^2D_1^2D_3D_4\beta^2\gamma\delta \ -kmD_1D_3^2D_4\alpha\beta^2\delta$$

$$+klD_1D_2D_3D_4\alpha\beta\gamma\delta \quad -klD_1D_2D_3D_4\alpha\beta\gamma\delta \quad +kmD_1D_3^2D_4\alpha\beta(\beta-1)\delta \quad +klD_1D_2D_3D_4\alpha\beta\gamma\delta$$

$$-klD_1D_2D_3D_4\alpha\beta\gamma\delta \quad +lmD_2D_3^2D_4\alpha(\alpha-1)\beta\delta \quad -l^2D_2^2D_3D_4\alpha(\alpha-1)\gamma\delta \quad +l^2D_2^2D_3D_4\alpha^2\gamma\delta$$

$$-lmD_2D_3^2D_4\alpha^2\beta\delta \quad -knD_1D_3D_4^2\alpha\beta^2\gamma \quad +knD_1D_3D_4^2\alpha\beta(\beta-1)\gamma \quad -mnD_3^2D_4^2\alpha(\alpha-1)\beta(\beta-1)$$

$$+nlD_2D_3D_4^2\alpha(\alpha-1)\gamma\delta -nlD_2D_3D_4^2\alpha^2\gamma\delta +mnD_3^2D_4^2\alpha^2\beta^2\Big\}$$

$$=-\frac{D_4-1}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}}{D_1^2D_2^2D_3D_4}\Big\{-kD_1\beta(\delta-1) \ -kD_1(\beta-1)\delta \ +kD_1(\beta-1)(\delta-1) \ +kD_1\beta\delta$$

$$+lD_2(\alpha-1)(\delta-1) \ -lD_2\alpha(\delta-1) \ +lD_2\alpha\delta \ -lD_2(\alpha-1)\delta \ -2mD_3\alpha\beta\gamma^{-1}\delta \ +mD_3\alpha(\beta-1)\gamma^{-1}\delta$$

$$-mD_3(\alpha-1)(\beta-1)\gamma^{-1}(\delta-1) \quad +mD_3\alpha\beta\gamma^{-1}(\delta-1) \quad +mD_3(\alpha-1)\beta\gamma^{-1}\delta \quad +nD_4(\alpha-1)(\beta-1)$$

$$-nD_4(\alpha-1)\beta \quad +nD_4\alpha\beta \quad -nD_4\alpha(\beta-1)\Big\} \quad +\frac{(1-2D_4)}{|J|}\frac{A^2D_1^{2\alpha}D_2^{2\beta}D_3^{2\gamma}D_4^{2\delta}}{D_1^2D_2^2D_3D_4} \quad \frac{AD_1^{\alpha}D_2^{\beta}D_3^{\gamma}D_4^{\delta}\Re}{B}$$

$$\Big\{-k^2D_1^2\beta(\beta-1)\gamma\delta \quad +k^2D_1^2\beta^2\gamma\delta \quad -kmD_1D_3\alpha\beta^2\delta \quad +kmD_1D_3\alpha\beta(\beta-1)\delta \quad -lmD_2D_3\alpha^2\beta\delta$$

$$+lmD_2D_3\alpha(\alpha-1)\beta\delta \quad -l^2D_2^2\alpha(\alpha-1)\gamma\delta \quad +l^2D_2^2\alpha^2\gamma\delta \quad -knD_1D_4\alpha\beta^2\gamma \quad +knD_1D_4\alpha\beta(\beta-1)\gamma$$

$$+nlD_2D_4\alpha(\alpha-1)\beta\gamma-nlD_2D_4\alpha^2\beta\gamma \ -mnD_3D_4\alpha(\alpha-1)\beta(\beta-1)+mnD_3D_4\alpha^2\beta^2\Big\}$$

$$=-\frac{D_4-1}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}B}{D_1^2D_2^2D_3D_4\Re}\Big\{-\alpha\beta(\delta-1) \quad +\alpha(\beta-1)(\delta-1) \quad +(\alpha-1)\beta(\delta-1)$$

$$-(\alpha-1)(\beta-1)(\delta-1) \quad +(2D_4-1)(\alpha-1)(\beta-1)\delta \quad -(2D_4-1)(\alpha-1)\beta\delta \quad +(2D_4-1)\alpha\beta\delta$$

$$-(2D_4-1)\alpha(\beta-1)\delta\Big\} \quad +\frac{(1-2D_4)}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}B}{D_1^2D_2^2D_3D_4\Re}\Big\{\alpha(\beta-1) \quad +(\alpha-1)\beta \quad -\alpha\beta$$

$$-(\alpha-1)(\beta-1)\Big\}$$

$$\frac{\partial D_3}{\partial n_0}=-\frac{1}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}B}{D_1^2D_2^2D_3D_4\Re}\Big\{2D_4^2\delta+D_4(3-4\delta)+2\delta-2\Big\}. \tag{32}$$

Now using $\alpha=\beta=\gamma=\delta=\frac{1}{4}$ then we get, $\Re=1$, i.e., for constant returns to scale, in (32) we get,

$$\frac{\partial D_3}{\partial n_0} = -\frac{1}{|J|} \frac{A^3 B}{2^9 D_1^{\frac{5}{4}} D_2^{\frac{5}{4}} D_3^{\frac{1}{4}} D_4^{\frac{1}{4}}} \left\{ (D_4 + 2)^2 - 10 \right\}. \tag{33}$$

If $D_4 > \left(\sqrt{10} - 2\right)$ in (33) we get,

$$\frac{\partial D_3}{\partial n_0} < 0 . \tag{34}$$

Inequality (34) shows that if the discounted price of the irregular raw material, n_0 increases; the level of principal raw material is decreased. The industry may face unsustainable circumstances for constant returns to scale, and irregular raw material related products should be reduced for the sustainability of the industry.

If $D_4 < \left(\sqrt{10} - 2\right)$ in (33) we get,

$$\frac{\partial D_3}{\partial n_0} > 0 . \tag{35}$$

Inequality (35) shows that if the discounted price of the irregular raw material, n_0 increases; the level of purchasing principal raw material also increases. It seems that irregular raw material is complementary to principal raw material.

If $D_4 = \left(\sqrt{10} - 2\right)$ in (33) we get,

$$\frac{\partial D_3}{\partial n_0} = 0 . \tag{36}$$

Equation (35) shows that if the discounted price of the irregular raw material, n_0 increases; the there is no change of purchasing principal raw material. It seems that there is no relation between principal and irregular raw materials.

Now we study the effect of irregular raw materials α_4 when if the discounted price of the irregular raw material, n_0 increases. Taking T_{54} (i.e., term of 5th row and 4th column) from both sides of (17) we get (Islam et al., 2011; Mohajan, 2017a; Mohajan & Mohajan, 2022c, 2023f),

$$\frac{\partial D_4}{\partial n_0} = -\frac{D_4^2}{|J|}[C_{15}] + \frac{D_4}{|J|}[C_{15}] - 2\lambda \frac{D_4}{|J|}[C_{55}] + \frac{\lambda}{|J|}[C_{55}]$$

$$= -\frac{D_4^2}{|J|}\text{Cofactor of } C_{15} + \frac{D_4}{|J|}\text{Cofactor of } C_{15} - 2\lambda \frac{D_4}{|J|}\text{Cofactor of } \mathrm{C}_{55} + \frac{\lambda}{|J|}\text{Cofactor of } \mathrm{C}_{55}$$

$$= \frac{D_4^2 - D_4}{|J|} \begin{vmatrix} -B_1 & T_{11} & T_{12} & T_{13} \\ -B_2 & T_{21} & T_{22} & T_{23} \\ -B_3 & T_{31} & T_{32} & T_{33} \\ -B_4 & T_{41} & T_{42} & T_{43} \end{vmatrix} + \frac{\lambda(1 - 2D_4)}{|J|} \begin{vmatrix} 0 & -B_1 & -B_2 & -B_3 \\ -B_1 & T_{11} & T_{12} & T_{13} \\ -B_2 & T_{21} & T_{22} & T_{23} \\ -B_3 & T_{31} & T_{32} & T_{33} \end{vmatrix}$$

$$= \frac{D_4^2 - D_4}{|J|} \left\{ -B_1 \begin{vmatrix} T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \\ T_{41} & T_{42} & T_{43} \end{vmatrix} - T_{11} \begin{vmatrix} -B_2 & T_{22} & T_{23} \\ -B_3 & T_{32} & T_{33} \\ -B_4 & T_{42} & T_{43} \end{vmatrix} + T_{12} \begin{vmatrix} -B_2 & T_{21} & T_{23} \\ -B_3 & T_{31} & T_{33} \\ -B_4 & T_{41} & T_{43} \end{vmatrix} - T_{13} \begin{vmatrix} -B_2 & T_{21} & T_{22} \\ -B_3 & T_{31} & T_{32} \\ -B_4 & T_{41} & T_{42} \end{vmatrix} \right\}$$

$$+\frac{\lambda(1-2D_4)}{|J|}\left\{B_1\begin{vmatrix}-B_1 & T_{12} & T_{13}\\ -B_2 & T_{22} & T_{23}\\ -B_3 & T_{32} & T_{33}\end{vmatrix}-B_2\begin{vmatrix}-B_1 & T_{11} & T_{13}\\ -B_2 & T_{21} & T_{23}\\ -B_3 & T_{31} & T_{33}\end{vmatrix}+B_3\begin{vmatrix}-B_1 & T_{11} & T_{12}\\ -B_2 & T_{21} & T_{22}\\ -B_3 & T_{31} & T_{32}\end{vmatrix}\right\}$$

$$=\frac{D_4^2-D_4}{|J|}\left[-B_1\{T_{21}(T_{32}T_{43}-T_{42}T_{33})+T_{22}(T_{41}T_{33}-T_{31}T_{43})+T_{23}(T_{31}T_{42}-T_{41}T_{32})\}\right.$$

$$-T_{11}\{-B_2(T_{32}T_{43}-T_{42}T_{33})+T_{22}(-B_4T_{33}+B_3T_{43})+T_{23}(-B_3T_{42}+B_4T_{32})\}$$

$$+T_{12}\{-B_2(T_{31}T_{43}-T_{41}T_{33})+T_{21}(-B_4T_{33}+B_3T_{43})+T_{23}(-B_3T_{41}+B_4T_{31})\}$$

$$\left.-T_{13}\{-B_2(T_{31}T_{42}-T_{41}T_{32})+T_{21}(-B_4T_{32}+B_3T_{42})+T_{22}(-B_3T_{41}+B_4T_{31})\}\right]$$

$$+\frac{\lambda(1-2D_4)}{|J|}\left[B_1\{-B_1(T_{22}T_{33}-T_{32}T_{23})+T_{12}(-B_3T_{23}+B_2T_{33})+T_{13}(-B_2T_{32}+B_3T_{22})\}\right.$$

$$-B_2\{-B_1(T_{21}T_{33}-T_{31}T_{23})+T_{11}(-B_3T_{23}+B_2T_{33})+T_{13}(-B_2T_{31}+B_3T_{21})\}$$

$$\left.+B_3\{-B_1(T_{21}T_{32}-T_{31}T_{22})+T_{11}(-B_3T_{22}+B_2T_{32})+T_{12}(-B_2T_{31}+B_3T_{21})\}\right]$$

$$=\frac{D_4^2-D_4}{|J|}\{-B_1T_{21}T_{32}T_{43}+B_1T_{21}T_{42}T_{33}-B_1T_{22}T_{41}T_{33}+B_1T_{22}T_{31}T_{43}-B_1T_{23}T_{31}T_{42}+B_1T_{23}T_{41}T_{32}$$

$$+B_2T_{11}T_{32}T_{43}-B_2T_{11}T_{42}T_{33}-B_4T_{11}T_{22}T_{33}+B_3T_{11}T_{22}T_{43}-B_3T_{11}T_{23}T_{42}+B_4T_{11}T_{23}T_{32}$$

$$-B_2T_{12}T_{31}T_{43}+B_2T_{12}T_{41}T_{33}-B_4T_{12}T_{21}T_{33}+B_3T_{12}T_{21}T_{43}-B_3T_{12}T_{23}T_{41}+B_4T_{12}T_{23}T_{31}$$

$$+B_2T_{13}T_{31}T_{42}-B_2T_{13}T_{41}T_{32}+B_4T_{13}T_{21}T_{32}-B_3T_{13}T_{21}T_{42}+B_3T_{13}T_{22}T_{41}-B_4T_{13}T_{22}T_{31}\}$$

$$+\frac{\lambda(1-2D_4)}{|J|}\{-B_1^2T_{22}T_{33}+B_1^2T_{32}T_{23}-B_1B_3T_{12}T_{23}+B_1B_2T_{12}T_{33}-B_1B_2T_{13}T_{32}+B_1B_3T_{13}T_{22}$$

$$+B_1B_2T_{21}T_{33}-B_1B_2T_{31}T_{23}+B_2B_3T_{11}T_{23}-B_2^2T_{11}T_{33}+B_2^2T_{13}T_{31}-B_2B_3T_{13}T_{21}$$

$$-B_1B_3T_{21}T_{32}+B_1B_3T_{31}T_{22}-B_3^2T_{11}T_{22}+B_2B_3T_{11}T_{32}-B_2B_3T_{12}T_{31}+B_3^2T_{12}T_{21}\}$$

$$=-\frac{D_4^2-D_4}{|J|}\frac{A^3D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}}{D_1^2D_2^2D_3^2D_4^2}\{-kD_1D_4\alpha\beta^2\gamma^2\delta \quad +kD_1D_4\alpha\beta^2\gamma(\gamma-1)\delta$$

$$-kD_1D_4\alpha\beta(\beta-1)\gamma(\gamma-1)\delta \quad +kD_1D_4\alpha\beta(y-1)\gamma(\gamma-1)\delta \quad -kD_1D_4\alpha\beta^2\gamma^2\delta \quad +kD_1D_4\alpha\beta^2\gamma^2\delta$$

$$+lD_2D_4\alpha(\alpha-1)\beta\gamma^2\delta \quad -lD_2D_4\alpha(\alpha-1)\beta\gamma(\gamma-1)\delta \quad -nD_4^2\alpha(\alpha-1)\beta(\beta-1)\gamma(\gamma-1)$$

$$+mD_3D_4\alpha(\alpha-1)\beta(\beta-1)\gamma\delta \quad -mD_3D_4\alpha(\alpha-1)\beta^2\gamma\delta \quad -nD_4^2\alpha(\alpha-1)\beta^2\gamma^2 \quad -lD_2D_4\alpha^2\beta\gamma^2\delta$$

$$+lD_2D_4\alpha^2\beta\gamma(\gamma-1)\delta \quad -nD_4^2\alpha^2\beta^2\gamma(\gamma-1) \quad +mD_3D_4\alpha^2\beta^2\gamma\delta \quad -mD_3D_4\alpha^2\beta^2\gamma\delta \quad +n\alpha_4^2x^2y^2z^2$$

$$+lD_2D_4\alpha^2\beta\gamma^2\delta \quad -lD_2D_4\alpha^2\beta\gamma^2\delta \quad +nD_4^2\alpha^2\beta^2\gamma^2 \quad -mD_3D_4\alpha^2\beta^2\gamma\delta \quad +mD_3D_4\alpha^2\beta(\beta-1)\gamma\delta$$

$$-nD_4^2\alpha^2\beta(\beta-1)\gamma^2\Big\} + \frac{(1-2D_4)}{|J|}\frac{A^2D_1^{2\alpha}D_2^{2\beta}D_3^{2\gamma}D_4^{2\delta}}{D_1^2D_2^2D_3^2D_4^2}\frac{AD_1^{\alpha}D_2^{\beta}D_3^{\gamma}D_4^{\delta}\Re}{B}\Big\{-k^2D_1^2D_4^2\beta(\beta-1)\gamma(\gamma-1)$$

$$+k^2D_1^2D_4^2\beta^2\gamma^2 - l^2D_2^2D_4^2\alpha(\alpha-1)\gamma(\gamma-1) + l^2D_2^2D_4^2\alpha^2\gamma^2 + m^2D_3^2D_4^2\alpha^2\beta^2 + klD_1D_2D_4^2\alpha\beta\gamma(\gamma-1)$$

$$-2klD_1D_2D_4^2\alpha\beta\gamma^2 \quad +klD_1D_2D_4^2\alpha\beta\gamma(\gamma-1) \quad -2kmD_1D_3D_4^2\alpha\beta^2\gamma \quad -kmD_1D_3D_4^2\alpha\beta^2\gamma$$

$$+kmD_1D_3D_4^2\alpha\beta(\beta-1)\gamma + kmD_1D_3D_4^2\alpha\beta(\beta-1)\gamma - m^2D_3^2D_4^2\alpha(\alpha-1)\beta(\beta-1)$$

$$+lmD_2D_3D_4^2\alpha(\alpha-1)\beta\gamma + lmD_2D_3D_4^2\alpha(\alpha-1)\beta\gamma - 2lmD_2D_3D_4^2\alpha^2\beta\gamma\Big\}$$

$$= -\frac{D_4-1}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}B}{D_1^2D_2^2D_3^2\Re}\Big\{-2\alpha\beta\gamma \quad +2\alpha\beta(\gamma-1) \quad +\alpha(\beta-1)\gamma \quad +(\alpha-1)(\beta-1)\gamma$$

$$-(\alpha-1)\beta\gamma \quad -(2D_4-1)(\alpha-1)\beta\gamma \quad -(2D_4-1)\alpha\beta(\gamma-1) \quad +(2D_4-1)\alpha\beta\gamma$$

$$-(2D_4-1)(\alpha-1)(\beta-1)(\gamma-1) \quad -(2D_4-1)\alpha(\beta-1)\gamma\Big\} \quad +\frac{(1-2D_4)}{|J|}\frac{A^3\alpha\beta\gamma D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}B}{D_1^2D_2^2D_3^2\Re}$$

$$\Big\{-\alpha(\beta-1)(\gamma-1) \quad -(\alpha-1)\beta(\gamma-1) \quad -(\alpha-1)(\beta-1)\gamma \quad +2\alpha\beta(\gamma-1) \quad -3\alpha\beta\gamma \quad +2\alpha(\beta-1)\gamma$$

$$+2(\alpha-1)\beta\gamma\Big\}$$

$$= -\frac{D_4-1}{|J|}\frac{A^3\alpha\beta\gamma\delta D_1^{3\alpha}D_2^{3\beta}D_3^{3\gamma}D_4^{3\delta}B}{D_1^2D_2^2D_3^2\Re}$$

$$\Big\{D_4(-2\alpha\beta\gamma+2\alpha+2\beta+2\gamma-2)+(2\alpha\beta\gamma-2\alpha\beta-2\alpha\gamma-\alpha-\beta+1)\Big\}$$

$$-\frac{(1-2D_4)}{|J|}\frac{A^2\alpha\beta\gamma D_1^{2\alpha}D_2^{2\beta}D_3^{2\gamma}D_4^{2\delta}B}{D_1^2D_2^2D_3^2\Re}(\alpha+\beta-\gamma). \tag{37}$$

Now using $\alpha=\beta=\gamma=\delta=\frac{1}{2}$ then we get, $\Re=2$, i.e., for increasing returns to scale, in (37) we get,

$$\frac{\partial D_4}{\partial n_0} = -\frac{1}{|J|}\frac{A^3\,D_4^{\frac{3}{2}}B}{2^7D_1^{\frac{1}{2}}D_2^{\frac{1}{2}}D_3^{\frac{1}{2}}}(D_4-1)(3D_4-7). \tag{38}$$

If $D_4<1$ or $D_4>\frac{7}{3}$ in (38) we get,

$$\frac{\partial D_4}{\partial n_0}<0. \tag{39}$$

Inequality (39) shows that if the discounted price of the irregular raw material, n_0 increases; the purchasing level of it decreases, which is reasonable. In this situation the industry may decrease the production of irregular raw material related products.

If $1 < D_4 < \frac{7}{3}$ in (38) we get,

$$\frac{\partial D_4}{\partial n_0} > 0. \tag{40}$$

Inequality (40) shows that if the discounted price of the irregular raw material, n_0 increases; the purchasing level of it also increases. It seems that irregular raw material is essential for the industry and it has no substitutes.

If $D_4 = 1$ or $D_4 = \frac{7}{3}$ in (38) we get,

$$\frac{\partial D_4}{\partial n_0} = 0. \tag{41}$$

Inequality (41) shows that if the discounted price of the irregular raw material, n_0 increases; there is no change of the purchasing level of it. It seems that the industry is indifferent about the discounted price of the irregular raw material.

11. Conclusions

In this study we have discussed the economic effects of various inputs of an industry when the discounted price of the irregular raw material is increased. Very few researchers consider nonlinear budget constraint in their study. But we have considered here nonlinear budget constraint to provide economic predictions through the profit maximization investigations. In this paper we have included Cobb-Douglas productions function as our profit function. We have used 5×5 bordered Hessian matrix and 5×5 Jacobian to operate the mathematical formulations.

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